SL(3|N) Wigner quantum oscillators: examples of ferromagnetic-like oscillators with noncommutative, square-commutative geometry

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Abstract. A system of N non-canonical dynamically free 3D harmonic oscillators is studied. The position and the momentum operators (PM-operators) of the system do not satisfy the canonical commutation relations (CCRs). Instead they obey the weaker postulates for the oscillator to be a Wigner quantum system. In particular the PM-operators fulfil the main postulate, which is due to Wigner: they satisfy the equations of motion (the Hamiltonian's equations) and the Heisenberg equations. One of the relevant features is that the coordinate (the momentum) operators do not commute, but instead their squares do commute. As a result the space structure of the basis states corresponds to pictures when each oscillating particle is measured to occupy with equal probability only finite number of points, typically the eight vertices of a parallelepiped. The state spaces are finite-dimensional, the spectrum of the energy is finite with equally spaced energy levels. An essentially new feature is that the angular momenta of all particles are aligned. Therefore there exists a strong interaction or correlation between the particles, which is not of dynamical, but of statistical origin. Another relevant feature is that the standard deviations of, say, the kth coordinate and the momenta of αth is $\Delta\hat{R}_{\alpha k}\Delta\hat{P}_{\alpha k} \leq p\hbar/|N-3|$ ($N \neq 3,$ $p-$fixed positive integer), namely instead of uncertainty relations one has "certainty" relations. The underlying Lie superalgebraic structure of the oscillator is also relevant and will be explained in the context.

1. Introduction

The title of the present paper comes to indicate from the very beginning that the geometry of the quantum system under consideration is noncommutative. To be more explicit, we study $N-$particle three-dimensional harmonic oscillators with a Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{N} \Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega^2}{2} \hat{\mathbf{R}}_\alpha^2 \Big), \tag{1.1}$$

such that the position operators $\hat{R}_{\alpha 1}, \hat{R}_{\alpha 2}, \hat{R}_{\alpha 3}$ do not commute with each other and the momentum operators $\hat{P}_{\alpha 1}, \hat{P}_{\alpha 2}, \hat{P}_{\alpha 3}$ do not commute, too:

$$[\hat{R}_{\alpha i}, \hat{R}_{\beta j}] \neq 0, \quad [\hat{P}_{\alpha i}, \hat{P}_{\beta j}] \neq 0, \quad i,j = 1,2,3. \;\; \alpha, \beta = 1,2,...,N. \tag{1.2}$$

For this reason the oscillator system is more involved to study. On the other hand however it is simpler, because the squares of all position and momentum operators (PM-operators) do commute with each other,

$$[\hat{R}_{\alpha i}^2, \hat{P}_{\beta j}^2] = 0, \quad [\hat{R}_{\alpha i}^2, \hat{R}_{\beta j}^2] = 0, \quad [\hat{P}_{\alpha i}^2, \hat{P}_{\beta j}^2] = 0, \quad i,j = 1,2,3. \;\; \alpha, \beta = 1,2,...,N, \tag{1.3}$$

which is not the case for the canonical oscillator.

The motivation for such "commutation" relations will be clear soon. Here we remark only that the above relations are not postulated. We derive them.

The other part of the title, *ferromagnetic-like oscillators*, is to stress on the very strong statistical interaction between the angular momentums of the oscillating particles. Despite of the lack of any dynamical interaction term in the Hamiltonian, all angular momentums of the particles are aligned, they point into one and the same direction, similar to the spins in ferromagnets. In the present case however the particles are spinless and they carry no electric charge.

Three other appropriate "candidates" for a place in the title were:

1. *"A Lie superalgebraic approach to quantum statistics"*, coming to indicate that with each infinite class of basic Lie superalgebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$ one can associate quantum statistics, namely particular for this class "commutation" relations between the position and the momentum operators (PM-operators). In this terminology the Bose and the Fermi statistics are $\mathcal{B}$ statistics, whereas the statistics of the present model is, as we shall see, $\mathcal{A}$ statistics and more precisely $sl(3|N)$ statistics.

2. *"Discrete quantum systems"* pointing out that (as a rule) the space structure of most of the basis states corresponds to a picture when the oscillating particle is measured to occupy with equal probability only finite number of points, typically the eight vertices of a parallelepiped (see figure 1, p. 35). This is another strong correlation property, because in most cases these points are the same for all particles.

3. *"Finite-level quantum systems"*, a small subtitle indicating that the results of the present paper can be of interest also in the context of quantum computing.

There are several other properties, which differ from those of canonical oscillators with Hamiltonian (1.1). Some of them are evident as for instance that there exist neither coordinate nor momentum representation. Other properties are not so evident. One of them, which will be derived in the present paper, is the analogue (or, rather, an "anti"-analogue) of the uncertainty relations. It reads:

$$\Delta\hat{R}_{\alpha k}\Delta\hat{P}_{\alpha k} \leq \frac{p\hbar}{|N-3|}, \quad N \neq 3, \tag{1.4}$$

where p is a fixed positive integer, labelling the state space under consideration. The above inequality holds simultaneously for any particle α and any coordinate k. Note that contrary to the canonical case here the left hand side is smaller than the right hand side.

Passing to a more systematic exposition, we recall the definition of a Wigner quantum system. It is based on the following six postulates [1] (in the Heisenberg picture for definiteness; the very name WQS was introduced in [2]):

(P1) The state space W is a Hilbert space. To every state of the system there corresponds a normed to 1 vector from W.

(P2) To every physical observable L there corresponds a Hermitian (self-adjoint and hence linear) operator $\hat{L}$ in W.

(P3) Given a physical observable L, the measurement outcome values it may assume, are just the eigenvalues of the operator $\hat{L}$.

(P4) The expectation value of L in a state ψ is given by $\langle\hat{L}\rangle_\Psi = (\psi, \hat{L}\psi)$.

(P5) $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_N$ are solutions of the equations of motion

(the Hamilton's equations), which for the Hamiltonian (1.1) read:

$$\dot{\hat{\mathbf{P}}}_\alpha = -m_\alpha \omega^2 \hat{\mathbf{R}}_\alpha, \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{1}{m_\alpha}\hat{\mathbf{P}}_\alpha, \quad \text{for} \quad \alpha = 1, \ldots, N. \tag{1.5}$$

(P6) $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_N$ are solutions of the Heisenberg equations

$$\dot{\hat{\mathbf{P}}}_\alpha = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{\mathbf{P}}_\alpha], \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{\mathbf{R}}_\alpha], \quad \text{for} \quad \alpha = 1, \ldots, N. \tag{1.6}$$

A WQS with harmonic oscillator Hamiltonian (and in particular the Hamiltonian (1.1)) is said also to be (N-particle 3D) *Wigner quantum oscillator* (WQO) or simply *Wigner oscillator*.

The only difference between the above postulates and the postulates of conventional quantum mechanics is that in the latter the postulate (P6) is replaced with the

($\tilde{P}6$) $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_N$, satisfy the canonical commutation relations (CCRs) (see [3], we wrote them as given in [4]):

$$[\hat{R}_{\alpha i}, \hat{P}_{\beta j}] = \mathrm{i}\hbar\delta_{\alpha i,\beta j}, \quad [\hat{R}_{\alpha i}, \hat{R}_{\beta j}] = [\hat{P}_{\alpha i}, \hat{P}_{\beta j}] = 0, \quad i,j = 1,2,3. \;\; \alpha, \beta = 1,2,...,N. \tag{1.7}$$

Postulates (P1) - (P5) remain the same.

A necessary step on the way to establish whether a given Hamiltonian admits an alternative statistics, i.e. admits alternative "commutation relations" between PM-operators, is to solve the so called *Wigner's problem*, namely to find out whether the postulates (P5) and (P6) admit common noncanonical solutions. In our case this means to solve the compatibility conditions

$$[\hat{H}, \hat{\mathbf{P}}_\alpha] = \mathrm{i}\hbar m_\alpha \omega^2 \hat{\mathbf{R}}_\alpha, \quad [\hat{H}, \hat{\mathbf{R}}_\alpha] = -\frac{\mathrm{i}\hbar}{m_\alpha}\hat{\mathbf{P}}_\alpha, \quad \text{for} \quad \alpha = 1, \ldots, N. \tag{1.8}$$

Our main task is to find (noncanonical) solutions of the above equations. We shall come back to this problem in the next section. At this place we postpone in order to say a few words as a justification for the definition of a WQS given above. In doing so one has to answer to at least three questions:

(a) Why to replace the postulate ($\tilde{P}6$) with (P6)?

(b) If so, does one obtain in this way new, different from (1.7), relations among the position and momentum operators?

(c) Do the new relations lead to new and interesting predictions?

The first two questions were actually raised and answered by Wigner in a two page publication [5] in 1950. First of all Wigner gave a positive answer to question (b). To this end he considered an example of a one-dimensional harmonic oscillator with a Hamiltonian ($m = \omega = \hbar = 1$) $\hat{H} = \frac{1}{2}(\hat{P}^2 + \hat{R}^2)$. Abandoning the canonical commutation relations (CCRs) $[\hat{P}, \hat{R}] = -i$, Wigner was searching for all operators $\hat{R}$ and $\hat{P}$, such that the "classical" equations of motion $\dot{\hat{P}} = -\hat{R}$, $\dot{\hat{R}} = \hat{P}$ were identical with the Heisenberg equations $\dot{\hat{P}} = -i[\hat{P}, H]$, $\dot{\hat{R}} = -i[\hat{R}, H]$. The result: Wigner found infinitely many solutions labelled by one positive integer $p = 1, 2, ...$ Only the $p = 1$ solution coincided with the canonical $\hat{R}$ and $\hat{P}$. In other words Wigner has shown that *the CCRs can be viewed as sufficient, but not necessary conditions for the Hamiltonian equations and Heisenberg equations to hold simultaneously.* The canonical postulates (P1) - (P5) and ($\tilde{P}6$) imply that any conventional quantum system is a Wigner quantum system, but they do not exhaust all WQSs.

Next Wigner answered also the question (a) noting that the Heisenberg equations (1.6) and the Hamilton's equations (1.5) have a more immediate physical significance than the CCRs. Therefore it is logically justified to postulate from the very beginning these equations instead of the CCRs (1.7).

Turning to question (c) we list some of the characteristics of WQSs studied so far. WQSs from the class $\mathcal{A}$ [6] basic Lie superalgebras [1, 7-10]:

(i) The state space is finite-dimensional.

(ii) The spectrum of the energy is equally spaced but finite.

(iii) The geometry is noncommutative.

(iv) The spectrum of the position operators is finite.

Very different are the properties of the WQSs related to the LSs from the class $\mathcal{B}$. In particular for the WQS related to the LS $osp(3/2)$ from this class [11,12]:

(v) The orbital momentum of two spinless particles curling around each other can be 1/2. This would mean that the spin has a classical analogue.

(vi) The state space is infinite-dimensional.

Various aspects of Wigner's idea were studied by several authors from different points of view. Among the earlier papers we mention [13 - 18], but the subject is of interest also now

[19 - 27]. Here are in short some of the results. In [14] Schweber extended the conclusions of Wigner to QFT showing (on an example of a scalar field) that the field's commutation relations are also not defined uniquely. Okubo [17] related the different solutions of the Wigner's problem (different quantization) to the circumstance that different Lagrangian may lead to one and the same equation of motion. In [19] the Wigner's problem was solved for a magnetic dipole precessing in a magnetic field, thus demonstrating that the equations of motion can be compatible with the Heisenberg equations not only for potentials of oscillating type. It is particularly interesting also that the class of noncanonical solutions determined in [19] includes the deformed CCRs [28], thus indicating that the quantum deformations can be viewed also as generalizations of quantum statistics.

A strong "push" for studying further alternative commutation relations came also from the predictions of string theory that the geometry of the space becomes noncommutative at very small distances (see [29] for a survey and the references therein). To similar conclusions lead also various deformed models (most of them in the sense of quantum groups (see [30] for a review and the references therein). However, the idea itself was already suggested by Heisenberg in the late 1930's (as explained in [31]) and perhaps the first example of this kind was given by Snyder [32].

The paper is organized as follows. In the next sections we recall shortly where the idea for application of the Lie superalgebras in quantum statistics comes from. This section contains no new results.

In section 3 we outline the mathematical structure of the $sl(3|N)$ WQO. We identify the position and the momentum operators $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_N$ as odd operators in such a way that the linear span of these operators and their anticommutators close the Lie superalgebra $sl(3|N)$. Already in this chapter it becomes evident that the angular momentum operators poses unusual properties. Up to multiplicative constants they are the same for all particles and coincide with the angular momentum operators for the entire system. Mathematically this means that the projections of the angular momentum of the different particles are described by operators, which are equal in the sense of operators (up to multiplicative constants). Physically it corresponds to a picture with angular momentum alignment of all particles.

In section 4 the state spaces $V(N,p),\ p = 1, 2, 3, ...$ of the WQO are introduced. These are finite-dimensional subspaces of the infinite-dimensional Fock space $W(3|N)$, generated by three pairs of Bose creation and annihilation operators (CAOs) and N pairs of Fermi

CAOs. Each $V(N,p)$ carries an irreducible representation of $sl(3|N)$. The state spaces corresponding to different p carry inequivalent representation of the LS $sl(3|N)$. A somewhat unusual feature of this construction is that the Bose operators are considered as odd generators, the Fermi operators are even elements and the Bose operators anticommute with the Fermi operators [33]. This unconventional grading of the Bose and Fermi operators is not accidental. In this way $W(3|N)$ carries an infinite-dimensional irreducible representation of the orthosymplectic LS $osp(3|N)$.

In section 5 the energy spectrum of the entire system and of each of the oscillating particles is derived. The energy spectrum of the system is equidistant but finite. The Hamiltonian (1.1) has $min(N,p)+1$ equally spaced energy levels with a gap between neighboring levels $\omega\hbar$. The multiplicity of each such level is computed. The ground state is nondegenerate only in the case when $N=p$. In the case $N=1$, namely for one 3D WQO the energy levels are only two and if in addition $p=1$ they coincide with the first two energy levels of a canonical 3D oscillator.

The energy of each individual oscillator is an integral of motion. The energy levels are again equidistant, but this time the gap between neighboring levels is a fraction of $\omega\hbar$ and more precisely it is $\omega\hbar/|N-3|$. One of the unexpected features of the single particle energy is that in certain cases its ground energy can be zero (together with coordinates, momenta and angular momentum).

Section 6 is the biggest one. Here the space structure of each basis vector from the Fock space is analyzed. It is shown that certain states correspond to a picture when a particle can be measured to be with equal probability on every point of a sphere. There are also states with a particle distribution along two circles (figure 5, p. 69), but the typical picture is that the particle is measured to occupy with equal probability the eight vertexes of a parallelepiped (see figure 1, p. 34). For any Fock state the standard deviation of the particles along any direction is computed too.

In Section 7 the $so(3)$ structure of the state space $V(N,p)$ is clarified. Each such space is decomposed into irreducible $so(3)$ modules. As already mentioned, up to overall constants the projections of angular momentums are the same for all particles, which results in the angular momentum alignment. In this section the parity operator of each particle is introduced and the property that the nests of the basis states are occupied with equal probability is proved.

The last section 8 contains some concluding remarks.

Some abbreviations and notation:

$[x, y] = xy - yx$, $\{x, y\} = xy + yx$.
(φ, ψ) - the scalar product between the states φ and ψ;
$\mathbf{Z}_+$ - all nonnegative integers,
$\mathbf{N}$ - all positive integers,
WQS(s) - Wigner quantum system(s),
WQO(s) - Wigner quantum oscillator(s),
CAOs - creation and annihilation operators.
CCRs - canonical commutation relations
PM-operators - position and momentum operators
QM - quantum mechanics; QFT - quantum field theory
LA - Lie algebra, LAs - Lie algebras
LS - Lie superalgebra, LSs - Lie superalgebras

2. Lie (super)algebraic approach to quantum statistics

We have already indicated that the most general approach to determine the admissible commutation relations between the PM-operators would be to find all common solutions of the compatibility equations (1.8). This task is however very difficult. For this reason in addition to the requirement the PM-operators to satisfy eqs. (1.8) we shall require these operators to generate a Lie superalgebra (LS) from the class $\mathcal{A}$ and more precisely the LS $sl(3|N)$.

At this place one may ask why to introduce additional restrictions, postulating that the PM-operators generate a Lie superalgebra (LS)? And why a Lie superalgebra and not a Lie algebra or any other algebraic structure? One possible answer to this question would be to say that such an assumption was a good guess. And this would be not a wrong answer. There is however a deeper reason, which is based on two key observations (see also [33] for a more detailed discussions in the frame of both QM and QFT).

The first key observation belongs to Green. In 1953 he has shown that also the statistics of quantum field theory (QFT) can be generalized to what was later called parastatistics [34]. Green was also the first to realize that the infinitely many solutions found by Wigner in [5] are nothing but different inequivalent representations of one pair of para-Bose

(pB) creation and annihilation operators (CAOs) $B^{\pm}$ defined as

$$B^{\pm} = \frac{1}{\sqrt{2}}(\hat{R} \mp i\hat{P}) \quad \Leftrightarrow \quad \hat{R} = \frac{1}{\sqrt{2}}(B^{+} + B^{-}), \quad \hat{P} = \frac{\mathrm{i}}{\sqrt{2}}(B^{+} - B^{-}). \tag{2.1}$$

The generalization for the Hamiltonian (1.1) goes as follows. Introduce in place of the PM-operators new unknown operators

$$B_{\alpha k}^{\pm} = \left(\frac{m_{\alpha}\omega}{2\hbar}\right)^{1/2} \hat{R}_{\alpha k} \mp \mathrm{i}(2m_{\alpha}\omega\hbar)^{-1/2}\hat{P}_{\alpha k}, \tag{2.2}$$

In terms of these operators the Hamiltonian and the compatibility conditions (1.8) read (see also [7]):

$$\hat{H} = \frac{\omega\hbar}{2}\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\{B_{\alpha i}^{+}, B_{\alpha i}^{-}\}, \tag{2.3}$$

$$\sum_{\beta=1}^{N}\sum_{j=1}^{3}[\{B_{\beta j}^{+}, B_{\beta j}^{-}\}, B_{\alpha i}^{\pm}] = 2B_{\alpha i}^{\pm}, \quad i = 1, 2, 3, \quad \alpha = 1, 2, \ldots, N. \tag{2.4}$$

Postulate that the operators $B_{\alpha i}^{\pm}$ satisfy the triple relations

$$[\{B_{\alpha i}^{\xi}, B_{\beta j}^{\eta}\}, B_{\gamma k}^{\varepsilon}] = \delta_{ik}\delta_{\alpha\gamma}(\varepsilon - \xi)B_{\beta j}^{\eta} + \delta_{jk}\delta_{\beta\gamma}(\varepsilon - \eta)B_{\alpha i}^{\xi}). \tag{2.5}$$

It is straightforward to verify that the operators (2.5) satisfy the compatibility condition (2.4). Replacing the double indices with one index, $\alpha i \rightarrow I$, etc one rewrites (2.5) in the form:

$$[\{B_{I}^{\xi}, B_{J}^{\eta}\}, B_{K}^{\varepsilon}] = \delta_{IK}(\varepsilon - \xi)B_{J}^{\eta} + \delta_{JK}(\varepsilon - \eta)B_{I}^{\xi}), \quad \xi, \eta, \varepsilon = \pm. \tag{2.6}$$

By definition the operators $B_{I}^{\pm}$ are called para-Bose operators (pB-operators). Hence these operators yield a new possible statistics for the oscillator under consideration. The para-Bose (pB) operators were introduced by Green in QFT for quantization of integer spin fields [34].

In the same paper Green [34, 35] generalized the Fermi statistics to para-Fermi (pF) statistics. The defining relations for any N pairs of para-Fermi CAOs read:

$$[[F_{I}^{\xi}, F_{J}^{\eta}], F_{K}^{\varepsilon}] = \frac{1}{2}(\eta - \epsilon)^2\delta_{JK}F_{I}^{\xi} - \frac{1}{2}(\xi - \varepsilon)^2\delta_{IK}F_{J}^{\eta}, \quad \xi, \eta, \varepsilon = \pm. \tag{2.7}$$

Certainly the triple relations (2.6) and (2.7) are satisfied by Bose and Fermi operators, respectively.

The second key observation is that any n pairs of pF operators generate the orthogonal Lie algebra $so(2n+1) \equiv B_n$ [36, 37]. In fact the linear span of $F_1^{\pm}, ..., F_n^{\pm}$ and $[F_j^{\pm}, F_k^{\pm}]$, $j, k = 1, ..., n$, is already closed under further commutations as this is evident from (2.5).

The LA B_n belongs to the class B of simple Lie algebras. There are four infinite classes of simple Lie algebras A, B, C, D (in order to avoid confusion we denoted them with italic letters). Therefore the Fermi and the para-Fermi statistics can be called B-statistics. In [38] it was shown that to each such class there correspond statistics (perhaps more than one): $A-, B-, C-$ and $D-$statistics, which are appropriate for quantization of spinor fields (for $A-$statistics see [39]).

Similarly, if one considers the pB operators as odd elements, then the linear span of all pB operators B_i^{ξ}, $i = 1, ..., n$, and all of their anticommutators $\{B_i^{\xi}, B_j^{\eta}\}$ close a Lie superalgebra [40], which is isomorphic to the basic Lie superalgebra $osp(1/2n) \equiv B(0|n)$ [41] in the classification of Kac [6].

The LS $B(0/n)$ belongs to the class $\mathcal{B}$ of the basic Lie superalgebras [6]. Therefore the Bose and, more generally the pB statistics can be called $\mathcal{B}-$statistics. Also in this case there exist four infinite classes of basic Lie superalgebras, denoted as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, and again with each such class one can associate statistics. Since every Lie algebra is a Lie superalgebra (with no odd generators), the classes A, B, C, D are contained in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, respectively.

So far only the physical properties of $\mathcal{A}$ [1, 2, 7 - 10] and $\mathcal{B}$-statistics [11, 12] were studied in more detail. Very recently however large classes of new solutions of compatibility conditions (1.8) for all basic LSs were determined [42] based on the results on generalized quantum statistic previously defined in [43].

Motivated by the above results, in the present paper we study the properties of a new class of Wigner oscillators, which can be called *sl(3|N)-Wigner oscillators.* This is to indicate that the position and the momentum operators of the particles are odd generators of the Lie superalgebra $sl(3|N)$ (in appropriate $sl(3|N)-$modules) and generate it. But this will be the topic of the next section.

3. Sl(3|N) Oscillators. Representation independent results

In this section we perform the main step towards explicit construction of $sl(3|N)$-Wigner oscillators: we find common solutions of Eqs. (1.5), (1.6) and (1.8) with position and momentum operators which are odd elements in the LS $sl(3|N)$ and generate it.

As a first step we replace the unknown operators $\hat{R}_{\alpha k}$ and $\hat{P}_{\alpha k}$ with new unknown operators, writing down the time dependence explicitly:

$$
\begin{aligned}
E(t)_{k,\alpha+3} &= \sqrt{\frac{|N-3|m_\alpha\omega}{4\hbar}}\ \hat{R}(t)_{\alpha k} - \mathrm{i}\varepsilon\sqrt{\frac{|N-3|}{4m_\alpha\omega\hbar}}\ \hat{P}(t)_{\alpha k},\\
E(t)_{\alpha+3,k} &= \sqrt{\frac{|N-3|m_\alpha\omega}{4\hbar}}\ \hat{R}(t)_{\alpha k} + \mathrm{i}\varepsilon\sqrt{\frac{|N-3|}{4m_\alpha\omega\hbar}}\ \hat{P}(t)_{\alpha k}
\end{aligned}
\tag{3.1}
$$

where $k = 1, 2, 3$, $\alpha = 1, 2, ..., N$ and

$$
\varepsilon = +1 \quad \text{if} \quad N > 3 \quad \text{and} \quad \varepsilon = -1 \quad \text{if} \quad N = 1, 2. \tag{3.2}
$$

In terms of the new variables we have:

(a) Position and momentum operators

$$
\begin{aligned}
\hat{R}(t)_{\alpha k} &= \sqrt{\frac{\hbar}{|N-3|m_\alpha\omega}}\ (E(t)_{k,\alpha+3} + E(t)_{\alpha+3,k}),\\
\hat{P}(t)_{\alpha k} &= \mathrm{i}\varepsilon\sqrt{\frac{m_\alpha\omega\hbar}{|N-3|}}\ (E(t)_{k,\alpha+3} - E(t)_{\alpha+3,k}).
\end{aligned}
\tag{3.3}
$$

(b) Hamiltonian

$$
\hat{H} = \sum_{\alpha=1}^{N}\sum_{k=1}^{3}\frac{\omega\hbar}{|N-3|}\{E(t)_{k,\alpha+3}, E(t)_{\alpha+3,k}\}. \tag{3.4}
$$

(c) Hamiltonian equations

$$
\dot{E}(t)_{k,\alpha+3} = \mathrm{i}\varepsilon\omega E(t)_{k,\alpha+3}, \quad \dot{E}(t)_{\alpha+3,k} = -\mathrm{i}\varepsilon\omega E(t)_{\alpha+3,k}, \tag{3.5}
$$

(d) Heisenberg equations

$$
\dot{E}(t)_{k,\alpha+3} = \frac{\mathrm{i}\omega}{|N-3|}\sum_{\beta=1}^{N}\sum_{j=1}^{3}[\{E(t)_{j,\beta+3}, E(t)_{\beta+3,j}\}, E(t)_{k,\alpha+3}], \tag{3.6a}
$$

$$
\dot{E}(t)_{\alpha+3,k} = \frac{\mathrm{i}\omega}{|N-3|}\sum_{\beta=1}^{N}\sum_{j=1}^{3}[\{E(t)_{j,\beta+3}, E(t)_{\beta+3,j}\}, E(t)_{\alpha+3,k}], \tag{3.6b}
$$

(e) Compatibility conditions:

$$
E(t)_{k,\alpha+3} = \frac{1}{N-3}\sum_{\beta=1}^{N}\sum_{j=1}^{3}[\{E(t)_{j,\beta+3}, E(t)_{\beta+3,j}\}, E(t)_{k,\alpha+3}], \quad N \neq 3 \tag{3.7a}
$$

$$
E(t)_{\alpha+3,k} = -\frac{1}{N-3}\sum_{\beta=1}^{N}\sum_{j=1}^{3}[\{E(t)_{j,\beta+3}, E(t)_{\beta+3,j}\}, E(t)_{\alpha+3,k}], \quad N \neq 3 \tag{3.7b}
$$

The time dependence of $E(t)_{k,\alpha+3}$ and $E(t)_{\alpha+3,k}$ is evident from (3.5), despite of the fact that these operators are still unknown:

$$E(t)_{k,\alpha+3} = E_{k,\alpha+3}(0)\mathrm{e}^{\mathrm{i}\varepsilon\omega t}, \quad E_{\alpha+3,k}(t) = E_{\alpha+3,k}(0)\mathrm{e}^{-\mathrm{i}\varepsilon\omega t}. \tag{3.8}$$

From now on we set:

$$E_{k,\alpha+3}(0) \equiv E_{k,\alpha+3}, \quad E_{\alpha+3,k}(0) \equiv E_{\alpha+3,k}. \tag{3.9}$$

Then

$$\hat{R}_{\alpha k}(t) = \sqrt{\frac{\hbar}{|N-3|m_\alpha\omega}}\left(E_{k,\alpha+3}\mathrm{e}^{\mathrm{i}\varepsilon\omega t} + E_{\alpha+3,k}\mathrm{e}^{-\mathrm{i}\varepsilon\omega t}\right), \tag{3.10a}$$

$$\hat{P}_{\alpha k}(t) = \mathrm{i}\varepsilon\sqrt{\frac{m_\alpha\omega\hbar}{|N-3|}}\left(E_{k,\alpha+3}\mathrm{e}^{\mathrm{i}\varepsilon\omega t} - E_{\alpha+3,k}\mathrm{e}^{-\mathrm{i}\varepsilon\omega t}\right), \tag{3.10b}$$

whereas the Hamiltonian is time independent:

$$\hat{H} = \sum_{\alpha=1}^{N}\sum_{k=1}^{3}\frac{\omega\hbar}{|N-3|}\{E_{k,\alpha+3}, E_{\alpha+3,k}\}. \tag{3.11}$$

Let us underline that the above equations (3.3)-(3.7) were obtained from (1.1), (1.5) and (1.6) just as a result of change of variables (3.1). $E_{\alpha+3,k}$ and $E_{k,\alpha+3}$ are still unknown operators. As a first step now we find operators which satisfy Eqs. (3.7) and are odd generators of $sl(3|N)$.

For convenience we consider $sl(3|N)$ as a subalgebra of the general linear LS $gl(3|N)$. The latter is a complex linear space with a basis E_{AB}, $A, B = 1, 2, ..., N{+}3$. The Z_2-grading on $gl(3|N)$ is imposed from the requirement that

$$E_{iA}, \quad E_{Ai}, \quad i = 1,2,3, \quad A = 4,5,...,N+3 \;\; \text{are odd generators,} \tag{3.12a}$$

$$E_{ij}, \quad E_{AB}, \quad i,j = 1,2,3, \quad A,B = 4,5,...N+3 \;\; \text{are even generators.} \tag{3.12b}$$

The supercommutator on $gl(3|N)$, turning it into a LS, is a linear extension of the relations

$$[\![E_{AB}, E_{CD}]\!] = \delta_{BC}E_{AD} - (-1)^{deg(E_{AB})deg(E_{CD})}\delta_{AD}E_{CB}, \tag{3.13}$$

where

$$[\![E_{AB}, E_{CD}]\!] \equiv E_{AB}E_{CD} - (-1)^{deg(E_{AB})deg(E_{CD})}E_{CD}E_{AB}. \tag{3.14}$$

In (3.13) and (3.14) $A, B, C, D = 1, 2, ..., N + 3$.

The LS $sl(3|N)$ is a subalgebra of $gl(3|N)$:

$$sl(3|N) = \text{span}\Big(g_A E_{AA} - g_B E_{BB}, E_{CD} | C \neq D, A, B, C, D = 1, ..., N + 3\Big), \tag{3.15}$$

where

$$g_1 = g_2 = g_3 = 1, \quad g_4 = g_5 = ... = g_{N+3} = -1. \tag{3.16}$$

Then

$$\mathcal{H}' = \text{span}\Big(E_{AA} | A = 1, ..., N + 3\Big) \tag{3.17a}$$

and

$$\mathcal{H} = \text{span}\Big(g_A E_{AA} - g_B E_{BB} | A, B = 1, ..., N + 3\Big). \tag{3.17b}$$

are Cartan subalgebras of $gl(3|N)$ and $sl(3|N)$, respectively. These algebras are certainly commutative.

It is evident from (3.12b) that the even subalgebra of $gl(3|N)$ is the Lie algebra $gl(3) \oplus gl(N)$ with

$$gl(3) = span\{E_{ij} | i, j = 1, 2, 3\} \ \text{ and } \ gl(N) = span\{E_{AB} | A, B = 4, 5, ..., N + 3\}. \tag{3.18}$$

Coming back to the Wigner problem we observe that the generators $E_{k,\alpha+3}, E_{\alpha+3,k}$ with $k = 1, 2, 3, \ \alpha = 1, ..., N$ do satisfy the compatibility conditions (3.7). From now on we consider this particular solution, i.e., we identify the variables (3.7) with the odd generators (3.12a) of $sl(3|N)$. It is easy to verify that the linear span of $E_{k,\alpha+3}, E_{\alpha+3,k}$ and their anticommutators yield $sl(3|N)$. Hence, the position and the momentum operators are also odd generators (see (3.3)) and they generate $sl(3|N)$. It is straightforward to verify that the time dependent operators (3.10) yield a simultaneous solution of the Hamilton's equations (1.5) and the Heisenberg equations (1.6). Hence $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_N$ obey propositions (P5) and (P6).

Already now we can draw certain conclusions which are consequences only of the postulates (P5) and (P6). These conclusions are representation independent, they have to hold in every state space.

Using only the supercommutation relations (3.13) one derives from (3.11)

$$\begin{aligned}\hat{H} &= \frac{\hbar\omega}{|N-3|}\big(N \sum_{i=1}^{3} E_{ii} + 3 \sum_{A=4}^{N+3} E_{AA}\big) \\ &= \frac{\hbar\omega}{|N-3|}\big(NE_{11} + NE_{22} + NE_{33} + 3E_{44} + \ldots + 3E_{N+3,N+3}\big)\end{aligned} \tag{3.19}$$

and therefore the Hamiltonian is an element from the Cartan subalgebra $\mathcal{H}$ of $sl(3|N)$. So are the Hamiltonians $\hat{H}_\alpha$ of each individual particle,

$$\hat{H}_\alpha = \frac{\hat{\mathbf{P}}_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega^2}{2}\hat{\mathbf{R}}_\alpha^2 = \frac{\omega\hbar}{|N-3|}\big(E_{11}+E_{22}+E_{33}+3E_{\alpha+3,\alpha+3}\big), \quad \alpha = 1,...,N, \tag{3.20}$$

and therefore the energy of each individual particle is preserved in time. Eq. (3.20) follows from

$$\hat{R}_{\alpha i}^2 = \frac{\hbar}{|N-3|m_\alpha\omega}\ (E_{i,i}+E_{\alpha+3,\alpha+3}), \tag{3.21}$$

$$\hat{P}_{\alpha i}^2 = \frac{m_\alpha\omega\hbar}{|N-3|}\ (E_{i,i}+E_{\alpha+3,\alpha+3}), \tag{3.22}$$

$$\hat{\mathbf{R}}_\alpha^2 = \frac{\hbar}{|N-3|m_\alpha\omega}\ (E_{11}+E_{22}+E_{33}+3E_{\alpha+3,\alpha+3}), \tag{3.23}$$

$$\hat{\mathbf{P}}_\alpha^2 = \frac{m_\alpha\omega\hbar}{|N-3|}(E_{11}+E_{22}+E_{33}+3E_{\alpha+3,\alpha+3}). \tag{3.24}$$

In the above expressions $i = 1,2,3$ and $\alpha = 1,2,...,N$. The conclusion is that independently on the representation and hence in every state space the set of all operators

$$\hat{H},\ \hat{H}_\alpha,\ \hat{\mathbf{R}}_\alpha^2,\ \hat{\mathbf{P}}_\alpha^2,\ \hat{R}_{\alpha i}^2,\ \hat{P}_{\alpha i}^2, \quad i=1,2,3, \quad \alpha = 1,2,...,N, \tag{3.25}$$

constitute a commutative set of operators, whereas neither coordinates nor momenta commute, see (1.2). In this respect the WQO belongs to the class of models of non-commutative quantum oscillators [44 - 48] and, more generally, to theories with non-commutative geometry [49], [50]. Moreover, as in [51], the coordinates of the particles are observables with a quantized spectrum just like energy, angular momentum, etc.

This is an appropriate place to mention that any physical observable, which is a function of only even generators is time independent, which is another representation independent property. The latter stems from the observation that the even generators commute with the Hamiltonian,

$$[\hat{H}, E_{ij}] = 0 \ \text{ for any } \ i,j=1,2,3 \ \text{ or } \ i,j=4,5,...,N. \tag{3.26}$$

An important physical observable of this kind is the angular momentum. For the components of the orbital momentum $\hat{M}_{\alpha i}$ of each individual particle $\alpha = 1,2,..,N$, we postulate the same expression as in the canonical case:

$$\hat{M}_{\alpha i} = \frac{1}{2}\sum_{k,l=1}^{3} \varepsilon_{ikl}\{\hat{R}_{\alpha k}, \hat{P}_{\alpha l}\}. \tag{3.27}$$

Being an anticommutator of odd observables $\hat{R}_{\alpha i}$ and $\hat{P}_{\alpha i}$, each $\hat{M}_{\alpha i}$ is an even element and therefore it is an integral of motion, it does not depend on the time t. It is far from evident however that the above definition is physically acceptable. One necessary requirement is that for each α $\hat{M}_{\alpha 1},\ \hat{M}_{\alpha 2},\ \hat{M}_{\alpha 3}$ transform as vector operators under space rotations. The position and the momentum operators of each particle have to transform also as vectors with respect to space rotations. This is clear. But so far we do not have generators of the physical rotation group. In order to determine them we proceed also as in canonical quantum mechanics.

To begin with we express the projections of the angular momentum of each particle via Weyl generators:

$$\hat{M}_{\alpha i} = -\mathrm{i}\frac{\varepsilon\hbar}{|N-3|}\sum_{k,l=1}^{3}\varepsilon_{ikl}E_{kl} = -\mathrm{i}\frac{\hbar}{N-3}\sum_{k,l=1}^{3}\varepsilon_{ikl}E_{kl}. \tag{3.28}$$

Setting

$$\hat{M}_{\alpha i} = \frac{\hbar}{N-3}\hat{S}_{\alpha i} \tag{3.29}$$

we obtain:

$$\hat{S}_{\alpha i} = -\mathrm{i}\sum_{k,l=1}^{3}\varepsilon_{ikl}E_{kl}. \tag{3.30}$$

Explicitly

$$\hat{S}_{\alpha 1} = \mathrm{i}(E_{32}-E_{23}), \quad \hat{S}_{\alpha 2} = \mathrm{i}(E_{13}-E_{31}), \quad \hat{S}_{\alpha 3} = i(E_{21}-E_{12}) \tag{3.31}$$

with commutation relations

$$[\hat{S}_{\alpha j}, \hat{S}_{\alpha k}] = \mathrm{i}\sum_{l=1}^{3}\varepsilon_{jkl}\hat{S}_{\alpha l}, \tag{3.32}$$

which are the known commutation relations between the components of the angular momentum also in conventional QM. There is however one essential difference. In the canonical case the operators $\hat{S}_{\alpha 1}, \hat{S}_{\alpha 2}, \hat{S}_{\alpha 3}$, measure the components of the angular momentum of any particle in units $\hbar$. Here they are measured in units $\hbar/|N-3|$.

The most striking difference between the canonical oscillator (or the $sl(1|3N)$ oscillator [9, 10]) and the $sl(3|N)$ oscillator comes from the observation that the angular momentum operators $\hat{S}_{\alpha 1},\ \hat{S}_{\alpha 2},\ \hat{S}_{\alpha 3}$ do not depend on α. They are the same for all N particles (see the RHS of (3.30), (3.31)). In the next sections we will discuss this feature in more detail.

For the components of the angular momentum of the oscillator we have

$$\hat{M}_j = \sum_{\alpha=1}^{N} \hat{M}_{\alpha j} = \frac{\hbar N}{N-3}\big(-\mathrm{i} \sum_{k,l=1}^{3} \varepsilon_{jkl} E_{kl}\big) = \frac{\hbar N}{N-3}\hat{S}_j, \tag{3.33}$$

where

$$\hat{S}_j = -\mathrm{i} \sum_{k,l=1}^{3} \varepsilon_{jkl} E_{kl}, \tag{3.34}$$

or

$$\hat{S}_1 = \mathrm{i}(E_{32} - E_{23}), \quad \hat{S}_2 = \mathrm{i}(E_{13} - E_{31}), \quad \hat{S}_3 = \mathrm{i}(E_{21} - E_{12}). \tag{3.35}$$

In view of (3.33) $\hat{S}_j$ measures the components of the total angular momentum of the oscillator in units $\hbar N/(N-3)$:

$$[\hat{S}_j, \hat{S}_k] = \mathrm{i} \sum_{l=1}^{3} \varepsilon_{jkl} \hat{S}_l. \tag{3.36}$$

The operators

$$\hat{S}_+ = \hat{S}_1 + \mathrm{i}\hat{S}_2 = \mathrm{i}E_{32} - \mathrm{i}E_{23} - E_{13} + E_{31}, \tag{3.37}$$

$$\hat{S}_- = \hat{S}_1 - \mathrm{i}\hat{S}_2 = \mathrm{i}E_{32} - \mathrm{i}E_{23} + E_{13} - E_{31}, \tag{3.38}$$

$$\hat{S}_3 = \mathrm{i}(E_{21} - E_{12}), \tag{3.39}$$

satisfy the known commutation relations for the generators of the algebra $so(3)$.

$$[\hat{S}_3, \hat{S}_+] = \hat{S}_+, \quad [\hat{S}_3, \hat{S}_-] = -\hat{S}_-, \quad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_3. \tag{3.40}$$

At this place we postulate that $\hat{S}_1, \hat{S}_2, \hat{S}_3$ are the generators of space rotations. One verifies that

$$[\hat{S}_j, \hat{M}_{\alpha k}] = \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl} \hat{M}_{\alpha l}, \quad [\hat{S}_j, \hat{R}_{\alpha k}] = \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl} \hat{R}_{\alpha l}, \quad [\hat{S}_j, \hat{P}_{\alpha k}] = \mathrm{i}\sum_{l=1}^{3} \epsilon_{jkl} \hat{P}_{\alpha l}, \tag{3.41}$$

i.e., the components of $\hat{\mathbf{M}}_\alpha$, $\hat{\mathbf{R}}_\alpha$, $\hat{\mathbf{P}}_\alpha$ of each particle transform as vector operators with respect to space rotations. This holds in any representation. Moreover the total Hamiltonian and the Hamiltonians of each individual particle are scalars with respect to space rotations:

$$[\hat{S}_j, \hat{H}] = [\hat{S}_j, \hat{H}_\alpha] = 0, \quad j = 1, 2, 3, \quad \alpha = 1, 2, ..., N. \tag{3.42}$$

From the results obtained so far we can draw some further conclusions. First of all

$$[\hat{H}, \hat{P}^2_{\alpha i}] = [\hat{H}, \hat{R}^2_{\alpha i}] = 0. \tag{3.43}$$

The eigenvalues of $\hat{R}^2_{\alpha i}$ should be interpreted as squares of the admissible values for the ith coordinate of particle No α. The circumstance that $\hat{R}^2_{\alpha i}$ commutes with the Hamilonian then means that the square of the ith coordinate of the αth particle is an integral of motion. Since, moreover, all operators $\hat{P}^2_{\alpha i}$ and $\hat{R}^2_{\alpha i}$ commute with each other

$$[\hat{P}^2_{\alpha i}, \hat{P}^2_{\beta j}] = [\hat{P}^2_{\alpha i}, \hat{R}^2_{\beta j}] = [\hat{R}^2_{\alpha i}, \hat{R}^2_{\beta j}] = 0, \tag{3.44}$$

they can be measured simultaneously. Observe that the above statement is representation independent, it has to hold within every admissible state space.

4. Sl(3|N)-Wigner quantum oscillators. State spaces

So far we have introduced time dependent operators $\hat{\mathbf{R}}_1, \ldots, \hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_N$ which obey postulates (P5) and (P6). In the present section we determine state spaces, which are simultaneously representation spaces of $sl(3|N)$, so that the oscillator becomes a WQO. In principle one could search among all representations of $sl(3|N)$ and select those of them for which the Wigner problem has solutions. Since however explicit expressions for all representations are not available, we restrict our considerations to the class of ladder representations of $sl(3|N)$ [52] (leaving for future another class of known irreps, the essentially typical representations of $sl(3|N)$ [53]).

Below, following [52], we recall shortly the main properties of the ladder representations directly for $sl(3|N)$. Let

1. $\{c_1^\pm \equiv b_1^\pm,\ c_2^\pm \equiv b_2^\pm,\ c_3^\pm \equiv b_3^\pm\}$ be Bose operators considered as odd elements:

$$[b_i^-, b_j^+] = \delta_{ij}, \quad [b_i^+, b_j^+] = 0, \quad [b_i^-, b_j^-] = 0, \quad i,j = 1,2,3; \tag{4.1a}$$

2. $\{c_4^\pm \equiv f_4^\pm,\ c_5^\pm \equiv f_5^\pm, \ldots, c_{N+3}^\pm \equiv f_{N+3}^\pm\}$ be Fermi operators considered as even elements:

$$\{f_i^-, f_j^+\} = \delta_{ij}, \quad \{f_i^+, f_j^+\} = 0, \quad \{f_i^-, f_j^-\} = 0, \quad i,j = 4, ..., N+3 \qquad (4.1b);$$

3. The Bose operators anticommute with Fermi operators. (4.1c)

For a justification of this unusual grading see [54]. Here we only remark that with this grading the Fock space introduced below gives an infinite-dimensional irreducible representation of the orthosymplectic LS $B(m|n) \equiv osp(2m+1|2n)$ with generators the Bose and the Fermi operators and their supercommutators according to the grading.

Denote by $E(3|N)$ the Bose-Fermi algebra, namely the free superalgebra, generated by the CAOs $c_1^{\pm}, \ldots, c_{N+3}^{\pm}$ with the relations (4.1).

We are going to work in the Fock module $W(3|N)$ of $E(3|N)$ with an orthonormed basis

$$|n_1, n_2, \ldots, n_{N+3}\rangle = \frac{(c_1^+)^{n_1}(c_2^+)^{n_2}\ldots(c_{N+3}^+)^{n_{N+3}}}{\sqrt{n_1!n_2!n_3!}}|0\rangle, \quad c_k^-|0\rangle = 0. \tag{4.2}$$

where

$$n_1, n_2, n_3 \in \mathbf{Z}_+, \quad n_4, n_5, \ldots, n_{N+3} \in \{0, 1\}. \tag{4.3}$$

For definiteness n_1, n_2, n_3 are said to be *bosonic coordinates* of the state $|p;n\rangle$, whereas $n_4, ..., n_{N+3}$ are referred to as *fermionic coordinates* of $|p;n\rangle$. We shall see that each basis vector $|p;n\rangle$ determines up to a sign the possible coordinates of the particles. We call the basis (4.2) *a Fock basis.*

Clearly the representation of $E(3|N)$ in $W(3|N)$ is infinite dimensional. The transformations of the basis (4.2) under the action of the CAOs $c_i^{\pm}$ read ([52], Eq. (48)):

$$b_i^+|.., n_i, ..\rangle = \sqrt{n_i+1}|.., n_i+1, ..\rangle, \quad i = 1, 2, 3; \tag{4.4a}$$

$$b_i^-|.., n_i, ..\rangle = \sqrt{n_i}|.., n_i-1, ..\rangle, \quad i = 1, 2, 3; \tag{4.4b}$$

$$f_i^+|.., n_i, ..\rangle = (-1)^{n_1+..+n_{i-1}}\sqrt{1-n_i}|.., n_i+1, ..\rangle, \quad i = 4, 5, .., N+3; \tag{4.4c}$$

$$f_i^-|.., n_i, ..\rangle = (-1)^{n_1+..+n_{i-1}}\sqrt{n_i}|.., n_i-1, ..\rangle, \quad i = 4, 5, .., N+3; \tag{4.4d}$$

It follows ([52], (49), (50)):

$$c_i^+ c_j^- |.., n_i, .., n_j, ..\rangle = (g_i)^{n_1+..+n_{i-1}} (g_j)^{n_1+..+n_{j-1}} \sqrt{(1+g_i n_i)n_j} |.., n_i+1, .., n_j-1, ..\rangle,$$
$$\text{for} \quad i<j=1,2,...,N+3; \qquad (4.5a)$$

$$c_i^+ c_j^- |.., n_j, .., n_i, ..\rangle = (g_i)^{n_1+..+n_{i-1}-1} (g_j)^{n_1+..+n_{j-1}} \sqrt{(1+g_i n_i)n_j} |.., n_j-1, .., n_i+1, ..\rangle,$$
$$\text{for} \quad i>j=1,2,...,N+3; \qquad (4.5b)$$

$$c_i^+ c_i^- |.., n_i, ..\rangle = n_i |.., n_i, ..\rangle, \quad i=1,2,...,N+3. \qquad (4.5c)$$

Explicitly Eqs. (4.5) read:

$$b_i^+ b_j^- |.., n_i, .., n_j, ..\rangle = \sqrt{(n_i+1)n_j} |..n_i+1, .., n_j-1, ..\rangle, \quad i<j=1,2,3; \qquad (4.6a)$$
$$b_i^+ b_j^- |.., n_j, .., n_i, ..\rangle = \sqrt{(n_i+1)n_j} |..n_j-1, .., n_i+1, ..\rangle, \quad i>j=1,2,3; \qquad (4.6b)$$
$$f_i^+ f_j^- |.., n_i, .., n_j, ..\rangle = (-1)^{n_i+..+n_{j-1}} \sqrt{(1-n_i)n_j} |..n_i+1, .., n_j-1, ..\rangle,$$
$$j>i=4,5,...,N+3; \qquad (4.6c)$$
$$f_i^+ f_j^- |.., n_j, .., n_i, ..\rangle = (-1)^{n_j+..+n_{i-1}-1} \sqrt{(1-n_i)n_j} |..n_j-1, .., n_i+1, ..\rangle,$$
$$i>j=4,5,...,N+3; \qquad (4.6d)$$
$$b_i^+ f_j^- |.., n_i, .., n_j, ..\rangle = (-1)^{n_1+..+n_{j-1}} \sqrt{(n_i+1)n_j} |..n_i+1, .., n_j-1, ..\rangle,$$
$$i=1,2,3, \; j=4,5,...,N+3; \qquad (4.6e)$$
$$f_i^+ b_j^- |.., n_j, .., n_i, ..\rangle = (-1)^{n_1+..+n_{i-1}} \sqrt{(1-n_i)n_j} |..n_j-1, .., n_i+1, ..\rangle,$$
$$i=4,5...,N+3, \; j=1,2,3. \qquad (4.6f)$$

It is straightforward to verify that the following proposition holds [52].

Proposition 4.1: *The linear map defined on the generators as*

$$E_{ij} \longrightarrow c_i^+ c_j^-, \quad i,j=1,2,\ldots,N+3, \qquad (4.7)$$

gives a representation of the LS $gl(3|N)$ in $W(3|N)$.

From now on we write E_{ij} also for $c_i^+ c_j^-$, i.e., we use for simplicity one and the same symbol for the abstract generators of $gl(3|N)$ and for their images as operators in $W(3|N)$.

From Eqs. (4.5) one concludes that the infinite-dimensional Fock space $W(3|N)$ resolves into an infinite direct sum

$$W(3|N) = \bigoplus_{p=0}^{\infty} V(N,p), \tag{4.8}$$

of finite-dimensional subspaces

$$V(N,p) = span\{|n_1,...,n_{N+3}\rangle |\ n_1 + ... + n_{N+3} = p\}, \tag{4.9}$$

labelled with $p \in \mathbf{N}$ (all positive integers). We set

$$|n_1, n_2, n_3, n_4, ..., n_{N+3}\rangle \equiv |p; n_1, n_2, n_3, n_4, ..., n_{N+3}\rangle, \tag{4.10}$$

if we wish to underline that $|n_1,...,n_{N+3}\rangle \in V(N,p)$.

Each subspace $V(N,p)$ is invariant (and in fact irreducible) with respect to the operators (4.5) and hence with respect to any physical observable. Therefore each such subspace is a candidate for a state space of the system.

One verifies that the subspace

$$V(N,p,n_b,n_4,...,n_{N+3}) \subset V(N,p), \tag{4.11}$$

which is a linear span of all vectors $|n_1,...,n_{N+3}\rangle$ with $n_1+n_2+n_3 = n_b$ being fixed and $n_4,...,n_{N+3}$ being also fixed is an irreducible $gl(3)$ module. Similarly, the subspace

$$V(N,p,n_f,n_1,n_2,n_3) \subset V(N,p), \tag{4.12}$$

which is a linear span of all vectors $|n_1,...,n_{N+3}\rangle$ with both n_1,n_2,n_3 and $n_f = n_4+...+n_{N+3}$ fixed is an irreducible $gl(N)$ module. Finally, the subspace

$$V(N,p,n_b,n_f) \subset V(N,p), \tag{4.13}$$

which is a linear span of all vectors $|p;n_1,...,n_{N+3}\rangle$ with both $n_1+n_2+n_3 = n_b$ and $n_f = n_4+...+n_{N+3}$ is an irreducible $gl(3) \oplus gl(N)$ module. The labels n_b, n_f and p in (4.13) are not independent since $n_b + n_f = p$. Nevertheless we prefer to keep the more symmetrical notation in (4.13).

Proposition 4.2. *The $N-$particle 3D oscillator with PM-operators (3.10) and a state space $V(N,p)$ can be turned into a Wigner quantum oscillator for any $p = 1,2,...$.*

Proof.

(1). Every finite-dimensional linear space with a scalar product and in particular $V(N,p)$ is a Hilbert space. Postulating that to every state of the system there corresponds a normed to 1 vector from $V(N,p)$ one fulfills the first requirements (P1) of the definition of a WQS.

(2). Eqs (4.4) yield that the Hermitian conjugate of a creation operator is an annihilation operator:

$$(b_i^+)^* = b_i^-, \quad i = 1,2,3, \quad (f_A^+)^* = f_A^-, \quad A = 4,5,...,N+3. \tag{4.14}$$

Therefore

$$E_{ij}^* = (c_i^+ c_j^-)^* = (c_j^+ c_i^-) = E_{ji}, \quad i,j = 1,\ldots,N+3. \tag{4.15}$$

From (3.10) and (4.15) one concludes that the PM-operators are Hermitian operators in $V(N,p)$. As a consequence also the Hamiltonian (3.11), the Hamiltonians of each individual particle, the projections of the angular momentum $M_{\alpha j}$ and M_j are Hermitian operators. Hence the requirement (P2) holds too.

(3) The validity of (P5) and (P6) was already established in the previous section.

(4) Finally we postulate that any observable L can take only values which are eigenvalues of $\hat{L}$ (P3) and that the expectation value of L in a state ψ is evaluated according to (P4).

This completes the proof.

In this way to every $p = 1,2,...$ there corresponds an $sl(3|N)$ Wigner quantum oscillator with a state space $V(N,p)$. The PM-operators corresponding to different p are inequivalent because they correspond to different irreducible representations of $sl(3|N)$.

5. Physical properties - energy spectrum

In this section we begin to discuss the physical properties of the $sl(3|N)$ WQOs. We compute the energy spectrum of the system and of the individual oscillating particles. We shall see that the energy spectrum is equidistant, but finite. Another surprise is that in certain state spaces each individual particle can have a zero energy.

Let $V(N,p)$ be a p-state space, see (4.9). All vectors $|p; n_1, n_2, n_3; n_4, ..., n_{N+3}\rangle$ (with $n_1 + ... + n_{N+3} = p$) constitute a basis of eigenvectors of $\hat{H}$. For the eigenvalues of the

Hamiltonian (1.1) in this state one obtains from (3.19) and (4.5):

$$\hat{H}|p;n_1,...,n_{N+3}\rangle = \frac{\hbar\omega}{|N-3|}\Big(N\sum_{i=1}^{3} n_i + 3\sum_{A=4}^{N+3} n_A\Big)|p;n_1,...,n_{N+3}\rangle. \tag{5.1}$$

Therefore the energy $E(p;n_1,...,n_{N+3})$ of this state is

$$E(p;n_1,...,n_{N+3}) = \frac{\hbar\omega}{|N-3|}\big(N\sum_{i=1}^{3} n_i + 3\sum_{A=4}^{N+3} n_A\big). \tag{5.2}$$

Clearly all vectors $|p;n\rangle \equiv |p;n_1,...,n_{N+3}\rangle$ with one and the same

$$n_b = n_1 + n_2 + n_3 \ \text{ and } \ n_f = n_4 + ... + n_{N+3}, \tag{5.3}$$

have one and the same energy:

$$E(N,p,n_b,n_f) = \frac{\hbar\omega}{|N-3|}\big(Nn_b + 3n_f\big). \tag{5.4}$$

The energy $E(N,p,n_b,n_f)$ depends actually on three independent variables, for instance N,p,n_f since $n_b = p - n_f$.

Replace in the RHS of (5.4) n_b with $p - n_f$:

$$E(N,p,n_b,n_f) = \frac{\hbar\omega}{|N-3|}\big(Np - (N-3)n_f\big) = \omega\hbar\Big(\frac{Np}{|N-3|} - \frac{N-3}{|N-3|}n_f\Big). \tag{5.5}$$

Taking into account that

$$n_f = 0,1,2,...,\min(N,p) \Longleftrightarrow n_b = p, p-1, ..., \max(0, p-N), \tag{5.6}$$

one concludes:

Corollary 5.1 *The spectrum of the Hamiltonian $\hat{H}$ in $V(N,p)$ consists of*

$$d(N,p) \equiv \min(N,p) + 1 \tag{5.7}$$

equally spaced energy levels, with spacing $\omega\hbar$. The energy $E(N,p,n_b,n_f)$ corresponding to a particular value of n_f is (5.5).

Here comes the first big difference with a system of $N-$particle free canonical $3D$ oscillators. In the latter case the energy is also equally spaced with the same spacing $\hbar\omega$. Now however there is an infinite number of energy levels:

$$E_q = \hbar\omega\Big(\frac{3}{2}N + q\Big), \quad q = 0,1,2,.... \tag{5.8}$$

In order to determine the multiplicity of the energy $E(N, p, n_b, n_f)$ we have to compute the dimension of the eigenspace $V(N, p, n_b, n_f)$ of $\hat{H}$ corresponding to this eigenvalue.

By definition, see (4.13),

$$V(N, p, n_b, n_f) = span\{|p; n_1, ..., n_{N+3}\rangle \mid n_1 + n_2 + n_3 = n_b,\ n_b + n_f = p\}. \tag{5.9}$$

All states from $V(N, p, n_b, n_f)$ are stationary states, they have one and the same energy. Each subspace $V(N, p, n_b, n_f)$ is a $gl(3)\oplus gl(N)$ module. There are $d(N, p) = min(N, p)+1$ such modules,

$$V(N, p) = \sum_{n_f=0}^{min(N,p)} \oplus V(N, p, n_b, n_f), \quad n_b = p - n_f. \tag{5.10}$$

With respect to the even subalgebra each $V(N, p, n_b, n_f)$ behaves as a tensor product

$$V(N, p, n_b, n_f) = V_1(N, p, n_b) \otimes V_2(N, p, n_f), \quad n_b + n_f = p \tag{5.11}$$

of a $gl(3)-$module $V_1(N, p, n_b)$ and a $gl(N)-$module $V_2(N, p, n_f)$.

The linear space $V_1(N, p, n_b)$ is a Fock space of three Bose operators with a basis

$$|n_1, n_2, n_3\rangle = \frac{(c_1^+)^{n_1}(c_2^+)^{n_2}(c_3^+)^{n_3}}{\sqrt{n_1!n_2!n_3!}}|0\rangle, \quad c_k^-|0\rangle = 0, \quad n_1 + n_2 + n_3 = n_b = p - n_f. \tag{5.12}$$

We say that $V_1(N, p, n_b)$ is the *bosonic component* of $V(N, p, n_b, n_f)$ or a *bosonic subspace.* It is a simple exercise to verify that $V_1(N, p, n_b)$ is an irreducible $gl(3)$-module: there is only one eigenvector of the Cartan subalgebra of $gl(3)$ annihilated by the positive root vectors. This vector is the highest weight vector $|n_b, 0, 0\rangle$ with a highest weight $(n_b, 0, 0)$. The dimension of $V_1(N, p, n_b)$ is

$$\dim V_1(N, p, n_b) = (n_b + 1)(n_b + 2)/2. \tag{5.13}$$

Clearly the algebra $so(3)$ of the rotation group $SO(3)$ is a subalgebra of the algebra $gl(3)$ (see the expressions for the $so(3)$ generators(3.37) - (3.39)). Therefore the space rotations transform only the bosonic part $V_1(N, p, n_b)$ of $V(N, p, n_b, n_f)$.

The *fermionic subspace* $V_2(N, p, n_f)$ (the *fermionic part* of $V(N, p, n_b, n_f)$) has a basis

$$(f_4)^{n_4}(f_5)^{n_5}...(f_{N+3})^{n_{N+3}}|0\rangle, \quad n_4, n_5, \ldots, n_{N+3} \in \{0, 1\}, \quad n_4 + n_5 + ... + n_{N+3} = n_f.$$

It is also irreducible $gl(N)-$module with a highest weight vector $|1_1, 1_2, ...1_{n_f}, 0..., 0\rangle$ and dimension

$$\dim V_2(N, p, n_f) = \frac{N!}{n_f!(N - n_f)!}. \tag{5.14}$$

Therefore

$$\dim V(N,p,n_b,n_f) = \frac{(n_b+2)!}{2n_b!} \times \frac{N!}{n_f!(N-n_f)!}. \tag{5.15}$$

Observe the peculiarity of only one $3D$ Wigner oscillator ($N=1$): for any p the oscillator has only two energy levels, namely $E = \omega\hbar\frac{p}{2}$ and $E = \omega\hbar(\frac{p}{2}+1)$. In the case $p=1$ the result reduces to the first two energy levels of a canonical $3D$ oscillator.

Let us summarize.

Corollary 5.2 *The linearly independent states in $V(N,p)$ corresponding to the energy $E(N,p,n_b,n_f)$ are all vectors $|p;n_1,...,n_{N+3}\rangle$ with*

$$\sum_{i=1}^{3} n_i = n_b, \quad \sum_{A=4}^{N+3} n_A = n_f \tag{5.16}$$

(and $n_b+n_f=p$). Their number, and hence the multiplicity of $E(N,p,n_b,n_f)$ is given by Eq. (5.15).

For the ground energy $E(N,p)_{min}$ of the system one derives:

$$\text{If } N>3,\ N\geq p, \quad E(N,p)_{min} = \frac{\hbar\omega}{N-3}3p, \quad m = \frac{N!}{p!(N-p)!}, \tag{5.17a}$$

$$\text{If } N>3,\ N\leq p,\ E(N,p)_{min} = \frac{\hbar\omega}{N-3}N(p-N+3), \quad m = \frac{(p-N+2)!}{2(p-N)!}, \tag{5.17b}$$

$$\text{If } N<3, \quad E(N,p)_{min} = \frac{\hbar\omega}{3-N}Np, \quad m = \frac{1}{2}(p+2)(p+1), \tag{5.17c}$$

where m denotes the multiplicity of $E(N,p)_{min}$, the number of the linearly independent ground states. From the above results one concludes:

Corollary 5.3 *The ground energy of the system is always positive. In the cases $N>3$ the ground state is nondegenerate only if $N=p$.*

The circumstance that the energy of the ground states is never zero is not surprising. The same holds also in the canonical case. In conventional quantum mechanics also the energy of the ground state of each individual oscillator is positive. Is this the case for the WQOs? No, it is not. We proceed to show this.

Since the Hamiltonians $\hat{H}_1,...,\hat{H}_N$ of the oscillating particles are elements from the Cartan subalgebra of $sl(3|N)$, they commute with each other and with the Hamiltonian

of the system (1.1). Hence the energy of each individual particle is preserved in time. The energy of the α−th particle when the system is in the state $|p; n_1, ..., n_{N+3}\rangle$ is the eigenvalues of $\hat{H}_\alpha$ on this state. Since

$$\hat{H}_\alpha |p; n_1, ..., n_{N+3}\rangle = \frac{\hbar\omega}{|N-3|}(n_1 + n_2 + n_3 + 3n_{\alpha+3})|p; n_1, ..., n_{N+3}\rangle, \qquad (5.18)$$

the energy of the $\alpha - th$ particle in the state $|p; n_1, ..., n_{N+3}\rangle$ is

$$E_\alpha(p; n_1, ..., n_{N+3}) = \frac{\hbar\omega}{|N-3|}(n_1 + n_2 + n_3 + 3n_{\alpha+3}). \qquad (5.19)$$

Observe that in a given state $|p; n\rangle$ all particles can have at most two different energies: all particles No $\alpha_i, \alpha_j, ...$ with fermionic numbers $n_{\alpha_i} = n_{\alpha_j} = ... = 0$ have one and the same energy $E_0 = \frac{\hbar\omega}{|N-3|} n_b$, whereas the energy of the rest (those with fermionic numbers one) is $E_1 = \frac{\hbar\omega}{|N-3|}(n_b + 3)$.

As for the energy spectrum of each particle (after some combinatorics) one obtains:

Corollary 5.4. *The spectrum of the Hamiltonian H_α in $V(N,p)$, measured in units $\hbar\omega/|N-3|$, or, which is the same, the spectrum of the operator $E_{11}+E_{22}+E_{33}+3E_{\alpha+3,\alpha+3}$ (see (3.20)), reads*:

$$\{p - min(N,p), p - min(N,p) + 1, ..., p, p+1, p+2\}, \quad N > 3,\ p \geq 2; \qquad (5.20a)$$

$$\{0, 1, 3\}, \quad \text{for } N \geq 2, N \neq 3,\ p = 1; \qquad (5.20b)$$

$$\{p-1, p, p+1, p+2\}, \quad \text{for } N = 2,\ p \geq 2; \qquad (5.20c)$$

$$\{p, p+2\}, \quad \text{for } N = 1, p \geq 1; \qquad (5.20d).$$

In all cases with $p > 1$ and $N > 1$ there is a finite number of equally spaced energy levels, with spacing one (in units $\frac{\hbar\omega}{|N-3|}$). Peculiarities appear however in the state spaces with $p = 1$ or $N = 1$. In the case (5.20d) the spacing is twice bigger compared to $p > 1$ cases, whereas in the cases (5.20b) the equally spacing rule is violated.

6. Physical properties - oscillator configurations

Here we analyze the 3D space structure of the Fock states $|p; n\rangle$. Our results are based essentially on postulates (P1)-(P6). We begin however with a few observations of a

more general character. In particular we indicate that the *superposition principle* holds for any WQO. We point out also that the dispersion (or the square root of it - the standard deviation) is a convenient tool to search for particular states where the observables can have simultaneously particular values.

We are not going to discuss the details of the quantum measuring process and of how to prepare a system in a particular quantum state. Until now these are topics of increasingly hot discussions (see, for instance [55] and the references therein). We find however appropriate to recall the experimental definitions of certain entities and in particular of such in principle well known concepts as a mean value and a dispersion based on Gibbs ensemble (*gedanken*) experiments and to relate them with the theoretical predictions.

The definition to follow is not the most general one. It is adjusted directly for our considerations. Let Ψ be a particular state of the quantum system under consideration. Then the Gibbs ensemble (GE) corresponding to this particular state consists of a large number N_0 of identical quantum systems

$$\Psi^{(1)}, \Psi^{(2)}, \dots, \Psi^{(N_0)}, \tag{6.1}$$

all of them prepared to be in the state Ψ. By $\Psi^{(k)}$ we denote the kth individual such quantum system.

Consider the observable $\hat{L}$. Perform with all quantum systems from the GE simultaneously one and the same experiment, namely measure the value of the observable $\hat{L}$ in the state Ψ. Let $\tilde{n}_1$ quantum systems among all N_0 registered a value $\tilde{l}_1$, $\tilde{n}_2$ - a value $\tilde{l}_2$, and so on, $\sum_k \tilde{n}_k = N_0$. Then the average (=the mean = the expectation) value of $\hat{L}$ in the state Ψ is

$$\langle \hat{L} \rangle_\Psi^{exp} = \sum_k \frac{\tilde{n}_k}{N_0} \tilde{l}_k = \sum_k \tilde{P}_k \tilde{l}_k, \tag{6.2}$$

where $\tilde{P}_k = \tilde{n}_k / N_0$ is the probability to measure the result $\tilde{l}_k$. Note that by construction $\tilde{l}_1 \neq \tilde{l}_2 \neq ...$ It is assumed that the object under consideration admits a statistical description. This means that the values of the probabilities $\tilde{P}_k$ are practically independent on the number N_0 of the identical systems in the ensemble if this number is sufficiently large.

The first conclusion, which is due to postulate (P3), is that all numbers $\tilde{l}_1, \tilde{l}_2, ..$ have to be eigenvalues of the operator $\hat{L}$. Next, the "experimental" expectation value of $\hat{L}$ (6.2) should be consistent also with postulate (P4). Let $\Psi_1, ..., \Psi_g$ be an orthonormed system of eigenstates of $\hat{L}$: $\hat{L}\Psi_k = l_k \Psi_k, \ k = 1, 2...$ so that

$$\Psi = \alpha_1 \Psi_1 + \alpha_2 \Psi_2 + ... + \alpha_g \Psi_g. \tag{6.3a}$$

Then postulate (P4) yields:

$$\langle\hat{L}\rangle_{\Psi}^{theor} = (\Psi, \hat{L}\Psi) = \sum_{k=1}^{g} |\alpha_k|^2 l_k, \tag{6.3b}$$

This is actually the superposition principle for WQOs.

Superposition principle: *Let $\hat{L}$ be an observable and Ψ be a normed to one state, which is a linear combination $\Psi = \alpha_1\Psi_1 + \alpha_2\Psi_2 + ... + \alpha_g\Psi_g$, of an orthonormed set of eigenstates $\Psi_1, \Psi_2, \ldots, \Psi_g$ of $\hat{L}$: $\hat{L}\Psi_k = l_k\Psi_k$. Then*

$$\langle\hat{L}\rangle_{\Psi}^{theor} = (\Psi, \hat{L}\Psi\rangle = |\alpha_1|^2 l_1 + \ldots + |\alpha_n|^2 l_g, \tag{6.4}$$

where each coefficient $|\alpha_k|^2 = P_k$ gives the probability of measuring the eigenvalue l_k of $\hat{L}$ corresponding to the eigenstate Ψ_k.

A comparison of (6.2) with (6.4) gives that

$$\langle\hat{L}\rangle_{\Psi}^{exp} = \langle\hat{L}\rangle_{\Psi}^{theor} \equiv \langle\hat{L}\rangle_{\Psi}, \tag{6.5}$$

if $\tilde{P}_k$ is a sum of all $|\alpha_k^2|$ for which $l_k = \tilde{l}_k$. In particular if the spectrum of $\hat{L}$ is nondegenerate ($l_1 \neq l_2 \neq ...$), then $|\alpha_k^2| = \tilde{P}_k$.

If Ψ is an eigenstate of $\hat{L}$, $\hat{L}\Psi = l\Psi$, then the expectation value of $\hat{L}$ in the state Ψ yields the eigenvalue l, $\langle\hat{L}\rangle_{\Psi} = l$. The inverse is in general not true: from $\langle\hat{L}\rangle_{\Psi} = l$ it does not follow that $\hat{L}\Psi = l\Psi$. In other words from $\langle\hat{L}\rangle_{\Psi} = l$ one cannot conclude that the measured value of $\hat{L}$ in each single experiment is l. The entity which insures that all individual experiments measure one and the same eigenvalue for $\hat{L}$ in the state Ψ is the dispersion. The dispersion $\mathrm{Disp}(\hat{L})_{\Psi}$ of $\hat{L}$ in the state Ψ is by definition

$$\mathrm{Disp}(\hat{L})_{\Psi} = \sum_{k} \tilde{P}_k(\tilde{l}_k - \langle L\rangle_{\Psi})^2 \tag{6.6}$$

Then one verifies, taking into account the relation $|\alpha_k|^2 = P_k$, that

$$\mathrm{Disp}(\hat{L})_{\Psi} = \sum_{k=1}^{g} P_k(l_k - \langle L\rangle_{\Psi})^2 = \langle\hat{L}^2\rangle_{\Psi} - \langle\hat{L}\rangle_{\Psi}^2 = (\Psi, \hat{L}^2\Psi) - (\Psi, \hat{L}\Psi)^2. \tag{6.7}$$

As a consequence of (6.2) and (6.3) one has:

Proposition 6.1:*The next three statements are equivalent:*

(a) Ψ *is an eigenstate of the observable* $\hat{L}$*:* $\hat{L}\Psi = l\Psi$*,*
i.e. the observable $\hat{L}$ *has a definite (= a particular) value* l *in the state* Ψ*.*

(b) *The dispersion of the observable* $\hat{L}$ *in the quantum state* Ψ *is zero.*

(c) *All* N_0 *individual experiments from the GE register one and the same eigenvalue for* $\hat{L}$*.*

The generalization of proposition 6.1 to the case of any number of observebles requires some care.

Proposition 6.2. *The next three statements are equivalent:*

(a) Ψ *is simultaneously an eigenstate of each observables* $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_M$*:* $\hat{L}_k\Psi = l_k\Psi$*.*

(b) *The dispersion of all* $\hat{L}_1, \ldots, \hat{L}_M$ *in the quantum state* Ψ *is zero.*

(c) *In the state* Ψ *all observables* $\hat{L}_1, \ldots, \hat{L}_M$ *can be measured simultaneously in each individual experiment from the GE. In each such experiment they register one and the same eigenvalue* l_k *for each* $\hat{L}_k$*.*

The equivalence of (a) and (b) is evident from the very definitions (6.3a) and (6.6). To prove the part (c) it suffices to show that the measurement of any observable $\hat{L}_k$ does not disturb the simultaneous measurements of the rest of them, so that one can apply proposition 6.1 to any $\hat{L}_k$ separately. To this end we follow the considerations of Dirac [3] in QM. Without specifying the domain of definition of the operators, Dirac assumes that the simultaneous measurements of $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_M$ do not disturb each other if these operators mutually commute.

Let $D \subset W$ be the linear span of all common eigenvectors $\varphi_1, ..., \varphi_m$, of $\hat{L}_1, ..., \hat{L}_M$. The subspace D is invariant with respect to the action of $\hat{L}_1, ..., \hat{L}_M$. Moreover the operators $\hat{L}_1, ..., \hat{L}_M$, commute on D. Therefore, following the arguments of Dirac, we accept that the measurement of any observable $\hat{L}_k$, whenever the oscillator is in a state ϕ from D, does not disturb the simultaneous measurements of the other observables. In particular this holds for any eigenvector φ_k. Therefore one can apply proposition 6.1 for any observable $\hat{L}_k$ separately, thus proving proposition 6.2.

Coming back to the space structure of the oscillator, we recall that we work in a rectangular coordinate system. By

$$\mathbf{e} \equiv (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \tag{6.8}$$

we denote the three orthonormed frame vectors. Unless otherwise stated by coordinates of any 3D vector we have always in mind the coordinates with respect to this frame.

Our main goal in this section is to show that typically a state $|p;n\rangle$ can be interpreted as having a 3D-configuration with the property that one or more of the particles are allowed to occupy only a finite number of points.

For convenience we shall work not with the operators $\hat{R}_{\alpha k}(t)$ and $\hat{P}_{\alpha k}(t)$ themselves but with their dimensionless version

$$\hat{r}_{\alpha k}(t)=\sqrt{\frac{|N-3|m_\alpha\omega}{\hbar}}\hat{R}_{\alpha k}(t)=E_{k,\alpha+3}e^{i\varepsilon\omega t}+E_{\alpha+3,k}e^{-i\varepsilon\omega t}, \tag{6.9a}$$

$$\hat{p}_{\alpha k}(t)=\sqrt{\frac{|N-3|}{\hbar m_\alpha\omega}}\hat{P}_{\alpha k}(t)=i\varepsilon\Big(E_{k,\alpha+3}e^{i\varepsilon\omega t}-E_{\alpha+3,k}e^{-i\varepsilon\omega t}\Big). \tag{6.9b}$$

The peculiarity of the WQOs stems from the observation that the geometry of these oscillators is noncommutative:

$$[\hat{r}_{\alpha i},\hat{r}_{\beta j}]\neq 0, \qquad [\hat{p}_{\alpha i},\hat{p}_{\beta j}]\neq 0, \tag{6.10}$$

whereas in QM they do commute. On the other hand however all $6N$ operators $\hat{r}^2_{\alpha i},\ \hat{p}^2_{\beta j}$, $i,j=1,2,3,\ \alpha,\beta=1,2,...,N$, do commute with each other:

$$[\hat{r}^2_{\alpha i},\hat{r}^2_{\beta j}]=[\hat{r}^2_{\alpha i},\hat{p}^2_{\beta j}]=[\hat{p}^2_{\alpha i},\hat{p}^2_{\beta j}]=0, \tag{6.11}$$

whereas in QM they do not commute and more precisely $[\hat{r}^2_{\alpha i},\hat{p}^2_{\beta j}]\neq 0$. For this reason, as already mentioned in the introduction, we say that the geometry of the Wigner oscillator is noncommutative but *square commutative.*

Proposition 6.3. *The operators*

$$\hat{r}^2_{\alpha i}=E_{ii}+E_{\alpha+3,\alpha+3}, \qquad i=1,2,3, \qquad \alpha=1,2,...,N, \tag{6.12}$$

constitute a complete set of commuting operators in $V(N,p)$. In particular the eigenvalues $r^2_{\alpha i}$ of $\hat{r}^2_{\alpha i}$ determine uniquely the basis vectors $|p;n_1,n_2,n_3;n_4,...,n_{N+3}\rangle$

Proof. A given subspace $V(N,p)$ consists of the linear span of all vectors

$$|p;n_1,...,n_{N+3}\rangle \ \text{ with } \ n_1+n_2+n_3+n_4+...+n_{N+3}=p. \tag{6.13}$$

Any vector (6.13) is an eigenvector of $\hat{r}^2_{\alpha i}$:

$$\hat{r}^2_{\alpha i}|p; n_1, ..., n_{N+3}\rangle = \big(n_i + n_{\alpha+3}\big)|p; n_1, ..., n_{N+3}\rangle. \tag{6.14}$$

The claim is that all eigenvalues

$$r^2_{\alpha i} = n_i + n_{\alpha+3}, \quad i = 1, 2, 3, \quad \alpha = 1, 2, ..., N \tag{6.15}$$

of $\hat{r}^2_{\alpha i}$ determine uniquely the vector $|p; n_1, ..., n_{N+3}\rangle$.

Set

$$\sum_{A=4}^{N+3} r^2_{Ai} = a_i. \tag{6.16}$$

Then

$$\sum_{A=4}^{N+3} r^2_{Ai} = Nn_i + n_4 + n_5 + ... + n_{N+3} = a_i, \quad i = 1, 2, 3.$$

In order to eliminate $n_4, ..., n_{N+3}$ add in both sides of (6.16) $n_1 + n_2 + n_3$:

$$Nn_i + n_1 + n_2 + ... + n_{N+3} = a_i + n_1 + n_2 + n_3, \quad i = 1, 2, 3.$$

Taking into account (6.13) we obtain three equations for three unknown n_1, n_2, n_3 entities:

$$\begin{aligned}(N-1)n_1 - n_2 - n_3 &= a_1 - p,\\ -n_1 + (N-1)n_2 - n_3 &= a_2 - p,\\ -n_1 - n_2 + (N-1)n_3 &= a_2 - p.\end{aligned}$$

Their solution reads

$$\begin{aligned}n_1 &= \frac{(N-2)a_1 + a_2 + a_3 - pN}{N(N-3)},\\ n_2 &= \frac{(a_1 + (N-2)a_2 + a_3 - pN}{N(N-3)},\\ n_3 &= \frac{a_1 + a_2 + (N-2)a_3 - pN}{N(N-3)}\end{aligned} \tag{6.17}$$

The values for the other N entities $n_{\alpha+3}$, $\alpha = 1, 2, ..., N$ follow from (6.15):

$$n_{\alpha+3} = r^2_{i\alpha} - n_i, \quad i = 1, 2, 3, \quad \alpha = 1, 2, ..., N. \tag{6.18}$$

This completes the proof.

We proceed next to study the space configuration of the system whenever it is in a basis state $|p;n\rangle$. Any such state is simultaneously an eigenstate of all operators $\hat{r}^2_{\alpha i} = \hat{p}^2_{\alpha i} = E_{ii} + E_{\alpha+3,\alpha+3}$:

$$\hat{r}^2_{\alpha i}|p;..,n_i,...,n_{\alpha+3},..\rangle = r^2_{\alpha i}|p;..,n_i,...,n_{\alpha+3},..\rangle \tag{6.19}$$

and of the Hamiltonian. Therefore according to proposition 6.2 all these $6N$ operators can be measured simultaneously in each individual quantum system from the ensemble. The eigenvalue $r^2_{\alpha i}$ of $\hat{r}^2_{\alpha i}$ on $|p;n\rangle$ yields the square of the ith coordinate of particle α:

$$r^2_{\alpha i} = n_i + n_{\alpha+3}, \quad \alpha = 1,2,..,N, \quad i = x,y,z \text{ (or } 1,2,3). \tag{6.20}$$

From the last resut (6.20) one concludes.

Corollary 6.1. *A state $|p;n\rangle$ corresponds to a configuration when simultaneously for $k = 1,2,3$ the $k-th$ coordinate of αth particle is either $\sqrt{n_k + n_{\alpha+3}}$ or $-\sqrt{n_k + n_{\alpha+3}}$.*

What (6.20) does not say however is what is the probability the kth coordinate to be $\sqrt{n_k + n_{\alpha+3}}$ or $-\sqrt{n_k + n_{\alpha+3}}$. The next proposition answers this question too.

Proposition 6.4. *If the system is in the state $|p;n\rangle$, then with equal probability $1/2$ the first coordinate of particle α is (measured to be) either $\sqrt{n_1 + n_{\alpha+3}}$ or $-\sqrt{n_1 + n_{\alpha+3}}$, the second coordinate of the same particle is either $\sqrt{n_2 + n_{\alpha+3}}$ or $-\sqrt{n_2 + n_{\alpha+3}}$ and the third coordinate is either $\sqrt{n_3 + n_{\alpha+3}}$ or $-\sqrt{n_3 + n_{\alpha+3}}$. Also with probability $1/2$ the kth component of the momentum of particle α take values $\sqrt{n_k + n_{\alpha+3}}$ or $-\sqrt{n_k + n_{\alpha+3}}$, $k = 1,2,3$..*

Proof. The proof is based on the superposition principle and the explicit expressions for the eigenvectors of the coordinate operator $\hat{r}_{\alpha,k}$, $k = 1,2,3$. The latter read:

$a.$ Eigenvalue of $\hat{r}_{\alpha k}$: 0.

$$\text{Eigenstates}: \; v^0_{\alpha k}(..,0_k,..,0_{\alpha+3,..}) = |p;..,0_k,...,0_{\alpha+3},..\rangle, \tag{6.21a}$$

$b.$ Eigenvalues of $\hat{r}_{\alpha k}$: $\pm\sqrt{n_k}$ $(n_k \neq 0)$:

$$\text{Eigenstates}: \; v^{\pm}_{\alpha k}(..,n_k,..,0_{\alpha+3,..}) = \frac{1}{\sqrt{2}}\Big(|p;..,n_k,..,0_{\alpha+3},..\rangle$$
$$\mp (-1)^{n_1+...+n_{\alpha+2}} \mathrm{e}^{-\mathrm{i}\varepsilon\omega t}|p;..,n_k-1,..,1_{\alpha+3},..\rangle, \quad n_k > 0, \tag{6.21b}$$

where in place of the unwritten indices one inserts all admissible indices which are the same in the RHS and the LHS of one and the same equality. The vectors (6.21) constitute an orthonormed basis of eigenvectors of $\hat{r}_{\alpha k}(t)$ in $V(N,p)$ for any α and any k.

The inverse to (6.21b) relations take the form:

$$|p;..,n_k,,..,n_{\alpha+3},..\rangle = \frac{1}{\sqrt{2}}(-1)^{(n_1+\ldots+n_{\alpha+2}+1)n_{\alpha+3}}\mathrm{e}^{\mathrm{i}\varepsilon n_{\alpha+3}\omega t}$$
$$\Big(v_{\alpha k}^{-}(..,n_k+n_{\alpha+3},...,0_{\alpha+3},..)+(-1)^{n_{\alpha+3}}v_{\alpha k}^{+}(..,n_k+n_{\alpha+3},...,0_{\alpha+3},..)\Big), \quad (6.22)$$

Note that (for any $k=1,2,3$) the absolute values of the coefficients in front of $v_{\alpha k}^{+}$ and $v_{\alpha k}^{-}$ in (6.22) are equal and their square is 1/2. Then the superposition principle asserts that if $\hat{r}_k,\ k=1,2,3$ is measured, then with equal probability 1/2 the αth particle will be found to have a kth coordinate $\sqrt{n_k+n_{\alpha+3}}$ or $-\sqrt{n_k+n_{\alpha+3}}$, respectively.

The proof for the momentum is similar (see appendix E). This completes the proof.

Let us underline. For a given state $|p;n\rangle$ from $V(N,p)$ the interpretation of

$$r_{\alpha k} = \pm\sqrt{n_k+n_{\alpha+3}}, \quad k=1,2,3, \quad (6.23)$$

as coordinates of the particle α make sense only because $|r_{\alpha 1}|$, $|r_{\alpha 2}|$ and $|r_{\alpha 3}|$ are measured simultaneously in every individual experiment from the Gibbs ensemble. The circumstance that the coordinate operators $\hat{r}_{\alpha 1}$, $\hat{r}_{\alpha 2}$ and $\hat{r}_{\alpha 3}$ do not commute and therefore cannot have particular values is not used for the conclusion. The state $|p;n\rangle$ is anyhow not (and cannot be) an eigenstate of the coordinate operators. The conclusions are based on the fact that $|p;n\rangle$ is an eigenstate of the squares of the coordinate operators. The probability distribution for the coordinates, based on the superposition principle also does not contradict to the conclusions made so far.

We denote as

$$\Gamma(|p;n\rangle,\alpha) = \{\pm\sqrt{n_1+n_{\alpha+3}}\ \mathbf{e}_1 \pm \sqrt{n_2+n_{\alpha+3}}\ \mathbf{e}_2 \pm \sqrt{n_3+n_{\alpha+3}}\ \mathbf{e}_3\} \quad (6.24)$$

the positions where the αth particle can be measured to be (we say also "where the αth particle can be accommodated").

With equal probability 1/2 the αth particle will be measured to have a $k-$th coordinate $\sqrt{n_k+n_{\alpha+3}}$ or $-\sqrt{n_k+n_{\alpha+3}}$, $k=1,2,3$. Therefore the particle cannot be localized in only one of the points (6.23).

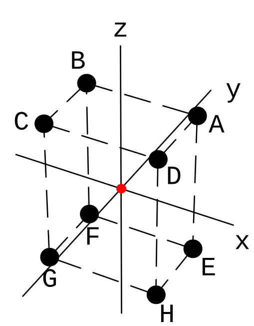
Following [9] and [10] we call the positions (6.24) nests. One should remember that the nests are not elements from the state space. They are just places where the particle can be measured to be. As we shall see, for certain states $|p;n\rangle$ the nests (6.24) do not exhaust all possible nests. In such a case the probabilities in proposition 6.4 are conditional probabilities.

In the remainder of this section we will concentrate mainly on the determination of all nests corresponding to a given state $|p;n\rangle$. On the way we derive an analogue of the uncertainty relations. We begin with an example.

Example 6.1. Let N=1. Consider a state $\varphi = |p=12;3,0,8;1\rangle$ and let the experiment measures simultaneously the absolute values of the coordinates x, y, z of the nests of the particle. Then each experiment from the Gibbs ensemble gives $|r_1| = 2$, $|r_2| = 1$, $|r_3| = 3$, which means that $r_1 = \pm 2$, $r_2 = \pm 1$, $r_3 = \pm 3$. Hence there are 8 nests where the first particle can be accommodated:

$$\Gamma(|p;3,0,8;1,...,0_{\alpha+3}\rangle,\ \alpha = 1) = \{\pm 2\mathbf{e}_1 \pm \mathbf{e}_2 \pm 3\mathbf{e}_3\}. \tag{6.25}$$

Figure 1



On Figure 1 we have given the space configuration of the first particle whenever the system is in the state $\varphi = |p;3,0,8;1,...\rangle$. The thick dots are the nests. There are 8 nests where the first particle can be accommodated. The coordinates of each vertex (= nest) are clear

from figure 1: $A = (2,1,3)$, $B = (-2,1,3)$, $C = (-2,-1,3)$, $D = (2,-1,3)$, etc. It is however impossible to predict which is the nest the particle is going to occupy in each individual experiment.

Let $P_X,\ X = A, B, C, D, E, F, G, H$, be the probability the first particle to be accommodated in the nest X. Then proposition 6.4 asserts that with equal probability 1/2 the particle will be measured to have a kth coordinate $\sqrt{n_k + n_4}$ and with the same probability the kth coordinate will be $-\sqrt{n_k + n_4}$. Therefore, see figure 1,

$$k = 1, \quad P_A + P_D + P_E + P_H = P_B + P_C + P_G + P_F = 1/2, \tag{6.26a}$$

$$k = 2, \quad P_A + P_B + P_F + P_E = P_C + P_D + P_G + P_H = 1/2, \tag{6.26b}$$

$$k = 3, \quad P_A + P_B + P_C + P_D = P_E + P_F + P_G + P_H = 1/2, \tag{6.26c}$$

$$P_A + P_B + P_C + P_D + P_E + P_F + P_G + P_H = 1. \tag{6.26d}$$

Clearly, equations (6.26) are not enough in order to determine all probabilities P_A,...,P_H. There are 7 equations for 8 undeterminate. We come back to this problem in section 7, where the equal probability $P_A = ... = P_H$ will be proved (see propositions 7.3 - 7.5).

The picture on Figure 1 rises many questions. *The main question* for us is whether $\Gamma(|p;n\rangle, \alpha)$ includes all nests where the α-th particle can be accommodated. Such a question make sense. The nests obtained so far are based on Eq. (6.19). Clearly, the corresponding nests can be only at the vertexes of a parallelepiped with each edge being parallel either to $\mathbf{e}_1$ or to $\mathbf{e}_2$ or to $\mathbf{e}_3$. This observation suggests to search for possible nests based on any other triad $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ of orthogonal unit vectors. Let $\hat{r}'_{\alpha 1}$, $\hat{r}'_{\alpha 2}$, $\hat{r}'_{\alpha 3}$ be the coordinate operators along $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$, respectively. We shall see that $(\hat{r}'_{\alpha 1})^2$, $(\hat{r}'_{\alpha 2})^2$, $(\hat{r}'_{\alpha 3})^2$ commute. If it happens in addition that

$$\hat{r}'^2_{\alpha i}|p; .., n_i, ..., n_{\alpha+3}, ..\rangle = r'^2_{\alpha i}|p; .., n_i, ..., n_{\alpha+3}, ..\rangle \tag{6.27}$$

then repeating the arguments from above, one would conclude that the points

$$\pm\sqrt{r'^2_{\alpha 1}}\ \mathbf{e}'_1,\ \pm\sqrt{r'^2_{\alpha 2}}\ \mathbf{e}'_2,\ \pm\sqrt{r'^2_{\alpha 3}}\ \mathbf{e}'_3, \quad i = 1, 2, 3, \tag{6.28}$$

are also nests for the αth particle. This time the nests would be (measured to be) in the vertexes of a parallelepiped, which edges are parallel to either $\mathbf{e}'_1$,, $\mathbf{e}'_2$, or $\mathbf{e}'_3$. Therefore, depending on the orientation of the triad, $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ the nests (6.28) could be new. We wish to underline that the result is certainly independent on the choice of the basis. The

nests (6.28) can be written in the basis (6.8), in the basis $\mathbf{e_1'}, \mathbf{e_2'}, \mathbf{e_3'}$ or in any other 3D basis. From a technical point of view however the choice of the basis may be relevant.

We find it convenient to consider all possible frames obtained by rotations from $\mathbf{e}$. Any such rotation is determined by a real orthogonal 3×3 matrix $g \equiv (g_{ij}),\ i, j = 1, 2, 3$, i.e., $gg^t = 1$ (t denotes a transposition) with determinant 1. The set of all such rotations constitute a group, the group $SO(3)$. To each g put in correspondence a linear operator T_g rotating the basis as follows: $T_g \mathbf{e}_k = (\mathbf{e}g)_k$. One verifies that the map $g \to T_g$ determines a representation of $SO(3)$ in 3D.

$$T_{g(1)} T_{g(2)} = T_{g(1)g(2)}, \quad T_E = E, \tag{6.29}$$

where E is the 3×3 unit matrix.

Denote by $\mathbf{e}(g) \equiv (\mathbf{e}(g)_1, \mathbf{e}(g)_2, \mathbf{e}(g)_3)$ the orthonormed frame obtained from the initial one (6.8) after a rotation g:

$$T_g e_k \equiv \mathbf{e}(g)_k = \sum_{i=1}^{3} \mathbf{e}_i g_{ik}, \ \ k = 1, 2, 3, \quad \text{or in matrix notation} \quad \mathbf{e}(g) = \mathbf{e}g. \tag{6.30}$$

We shall parameterize the matrices $g \in SO(3)$ (and hence the frames (6.30)) with the Euler angles α, β, γ as in [56]:

$$g(\alpha, \beta, \gamma) = \begin{pmatrix} \begin{array}{c|c|c} \cos\alpha\cos\beta\cos\gamma & -\cos\alpha\cos\beta\sin\gamma & \cos\alpha\sin\beta \\ -\sin\alpha\sin\gamma & -\sin\alpha\cos\gamma & \\ \hline \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\cos\beta\sin\gamma & \sin\alpha\sin\beta \\ +\cos\alpha\sin\gamma & +\cos\alpha\cos\gamma & \\ \hline -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{array} \end{pmatrix}. \tag{6.31}$$

The domain of definition of the Euler angles in (6.31) is

$$0 \leq \alpha < 2\pi, \quad 0 \leq \beta \leq \pi, \quad 0 \leq \gamma < 2\pi. \tag{6.32}$$

Different triples (α, β, γ) define different rotations apart from the cases $\beta = 0$ when $\alpha + \gamma = \alpha' + \gamma'$ defines one and the same rotation and $\beta = \pi$ when $\alpha - \gamma = \alpha' - \gamma'$ corresponds also to one and the same rotations around $\mathbf{e}_3$.

Each rotation $g(\alpha, \beta, \gamma)$ can be represented as a sequence of rotations around $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. One such possibility (which reduces to rotations only around $\mathbf{e}_2$ and $\mathbf{e}_3$) reads:

$$g(\alpha, \beta, \gamma) = g(\mathbf{e}_3, \alpha) g(\mathbf{e}_2, \beta) g(\mathbf{e}_3, \gamma), \tag{6.33}$$

where $g(\mathbf{e}_k, \varphi)$ is a rotation around $\mathbf{e}_k$ on angle φ. Explicitly

$$g(\mathbf{e}_1, \varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} = e^{-\mathrm{i}\varphi s_1}, \quad s_1 = \mathrm{i}(e_{32} - e_{23}), \tag{6.34a}$$

$$g(\mathbf{e}_2, \varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} = e^{-\mathrm{i}\varphi s_2}, \quad s_2 = \mathrm{i}(e_{13} - e_{31}), \tag{6.34b}$$

$$g(\mathbf{e}_3, \varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{-\mathrm{i}\varphi s_3}, \quad s_3 = \mathrm{i}(e_{21} - e_{12}). \tag{6.34c}$$

Here e_{ij} are the 3×3 matrix units. Hence $g(\alpha, \beta, \gamma)$ can be written also as

$$g(\alpha, \beta, \gamma) = e^{-i\alpha s_3} e^{-i\beta s_2} e^{-i\gamma s_3}. \tag{6.35}$$

Each state space $V(N,p)$ is an $sl(3|N)$ module and therefore it carries also a representation of the physical $SO(3)$ group with generators (of the $so(3)$ subalgebra)

$$\hat{S}_1 = \mathrm{i}(E_{32} - E_{23}), \quad \hat{S}_2 = \mathrm{i}(E_{13} - E_{31}), \quad \hat{S}_3 = \mathrm{i}(E_{21} - E_{12}).$$

To each rotation $g(\alpha, \beta, \gamma)$ there corresponds a "rotation" in the state space $V(N,p)$ by an unitary operator $\hat{U}(g(\alpha, \beta, \gamma))$:

$$\Phi \to \Phi' = \hat{U}(g(\alpha, \beta, \gamma))\Phi, \quad \Phi \in V(N,p). \tag{6.36}$$

where in agreement with (6.35)

$$\hat{U}(g(\alpha, \beta, \gamma) = e^{-i\alpha \hat{S}_3} e^{-i\beta \hat{S}_2} e^{-i\gamma \hat{S}_3}. \tag{6.37}$$

In particular the operators $\hat{U}(g(\mathbf{e}_k, \varphi))$, $k = 1, 2, 3$, corresponding to rotations $g(\mathbf{e}_k, \varphi)$ around $\mathbf{e}_k$ on angle φ read:

$$\hat{U}(g(\mathbf{e}_k, \varphi)) = e^{-i\varphi \hat{S}_k}, \ k = 1, 2, 3. \tag{6.38}$$

To each rotation g there corresponds also a "rotation" in the algebra of the observables, induced via the transformations $\hat{U}(g)$ of the state space. Indeed, consider the observable $\hat{L}$ and let Ψ be a normed to 1 state, which is a linear combination

$$\Psi = \alpha_1 \psi_1 + \alpha_2 \psi_2 + ... + \alpha_p \psi_p,$$

of an orthonormed set of eigenstates $\psi_1, \psi_2, \ldots, \psi_p$ of $\hat{L}$: $\hat{L}\psi_i = \lambda_i \psi_i$. Then

$$\langle \hat{L} \rangle_\Psi = (\Psi, \hat{L}\Psi) = |\alpha_1|^2 \lambda_1 + \ldots + |\alpha_n|^2 \lambda_n,$$

and the superposition principle asserts that $|\alpha_i|^2 = |(\Psi, \hat{L}\psi_i)|^2$ gives the probability of measuring the eigenvalue λ_i of the observable $\hat{L}$. Therefore this probability cannot depend on the choice of the coordinate frame. Based on this Wigner proved a stronger statement [57]: for any two states Φ and Ψ the matrix element $(\Phi, \hat{L}\Psi)$ should be invariant under any rotation g of the basis, namely

$$(\Phi, \hat{L}\Psi) = (\hat{U}(g)\Phi, \hat{L}(g)\hat{U}(g)\Psi). \tag{6.39}$$

Clearly this is the case only if

$$\hat{L}(g) = \hat{U}(g)\hat{L}\hat{U}(g)^{-1}. \tag{6.40}$$

For brevity we denote the above unitary transformation as $V(g)$:

$$V(g)\hat{L} = \hat{U}(g)\hat{L}\hat{U}(g)^{-1}. \tag{6.41}$$

Evidently

$$\hat{V}(g_1 g_2) = \hat{V}(g_1)\hat{V}(g_2), \quad \text{and} \quad \hat{V}(g^{-1}) = \hat{V}(g)^{-1}. \tag{6.42}$$

Hence the map $g \to \hat{V}(g)$ defines a representation of $SO(3)$ in the algebra of the observables (considered as a linear space).

Let now the operators under consideration be the coordinate operators for particle α: $\hat{\mathbf{r}}_\alpha = (\hat{r}_{\alpha 1}, \hat{r}_{\alpha 3}, \hat{r}_{\alpha 3})$. How do they transform under rotations?

Proposition 6.5. *The transformation relations of the operators* $(\hat{r}_{\alpha 1}, \hat{r}_{\alpha 2}, \hat{r}_{\alpha 3})$ *under global rotations* g *are the same as for the frame vectors:*

$$\hat{r}(g)_{\alpha i} = \hat{U}(g)\hat{r}_{\alpha i}\hat{U}(g)^{-1} = \sum_{j=1}^{3} \hat{r}_{\alpha j} g_{ji}, \quad i = 1, 2, 3, \tag{6.43}$$

or in a matrix form

$$\hat{\mathbf{r}}(g)_\alpha = (\hat{\mathbf{r}}_\alpha g). \tag{6.44}$$

For a proof see Appendix A.

The physical interpretation of $\hat{r}(g)_{\alpha k}$ is the same as in the canonical QM: $\hat{r}(g)_{\alpha,k}$ is the coordinate operator of the αth particle along $\mathbf{e}(g)_k$.

Since any unitary transformation preserves the commutation relations, from (6.43) one concludes that the operators $\hat{r}(g)^2_{\alpha k}$ commute:

$$[\hat{r}(g)^2_{\alpha i}, \hat{r}(g)^2_{\beta j}] = 0 \;\; \forall \;\; \alpha, \beta = 1, .., N, \quad i, j = 1, 2, 3. \tag{6.45}$$

Our next task is to determine the dispersion of $\hat{r}(g)^2_{\alpha,k}$ as a function of g whenever the system is in a fixed state $|p;n\rangle$, i.e., $\mathrm{Disp}(\hat{r}(g)^2_{\alpha k})_{|p;n\rangle}$. Then the matrices g for which the dispersion vanishes simultaneously for $k = 1, 2, 3$ will determine possible nests for the particle α under consideration.

From the transformation of the basis $|p;n\rangle$ under the action of $\hat{r}(g)_{\alpha,k}$

$$\begin{aligned} &\hat{r}(g)_{\alpha,k}|p; n_1, n_2, n_3; .., n_{\alpha+3}, ..\rangle = (-1)^{n_1+n_2+n_3+n_{\alpha+3}-1} \\ &\sum_{j=1}^{3} \Big(g_{jk} (e^{\mathrm{i}\varepsilon\omega t} \sqrt{(n_j+1) n_{\alpha+3}} |p; .., n_j + 1, ..; .., n_{\alpha+3} - 1, ..\rangle \\ &+ e^{-\mathrm{i}\varepsilon\omega t} \sqrt{(1 - n_{\alpha+3}) n_j} |p; .., n_j - 1, ..; .., n_{\alpha+3} + 1, ..\rangle) \Big) \end{aligned} \tag{6.46}$$

one concludes that the mean value of each coordinate operator $\hat{r}(g)_{\alpha,k}$ in the state $|p;n\rangle$ vanishes:

$$\langle \hat{r}(g)_{\alpha,k} \rangle_{|p;n\rangle} = 0. \tag{6.47}$$

Again from (6.46) and the circumstance that all $\hat{r}(g)_{\alpha,k}$ are Hermitian operators one computes the mean square deviation of each coordinate operator $\hat{r}(g)_{\alpha,k}$ in the state $|p;n\rangle$:

$$\langle \hat{r}(g)^2_{\alpha,k} \rangle_{|p;n\rangle} = (\hat{r}(g)_{\alpha,k}|p;n\rangle, \hat{r}(g)_{\alpha,k}|p;n\rangle) = \sum_{i=1}^{3} g^2_{ik}(n_i + n_{\alpha+3}), \quad k = 1, 2, 3. \tag{6.48}$$

In view of (6.47) the above relation yields actually the dispersion

$$\mathrm{Disp}(\hat{r}(g)_{\alpha,k})_{|p;n\rangle} = \sum_{i=1}^{3} g^2_{ik}(n_i + n_{\alpha+3}), \quad k = 1, 2, 3. \tag{6.49}$$

of $\hat{r}(g)_{\alpha,k}$ in the state $|p;n\rangle$.

Note that for a state $\varphi = |p;n\rangle$ such that $n_1 = n_2 = n_3 = n_{\alpha+3} = 0$ the dispersion (6.49) vanishes,

$$\mathrm{Disp}(\hat{r}(g)_{\alpha,k})_{\varphi} = 0. \tag{6.49a}$$

Moreover such a state always exists if $p < N - 3$.

We proceed to derive an analogue of the uncertainty relations for the position and the momentum operators. Choose an arbitrary direction in the 3D space determined by a unit vector $\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}(\theta, \varphi)$ with spherical coordinates θ and φ:

$$\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}(\theta, \varphi) = (\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\varphi). \tag{6.50}$$

Let g_0 be the rotation matrix (6.31) with Euler angles $\alpha = \varphi$, $\beta = \theta$ and an arbitrary $\gamma :\ g_0 = g_0(\varphi, \theta, \gamma)$. Then

$$\hat{\mathbf{n}}(\theta, \varphi) = \mathbf{e}(g_0)_3. \tag{6.51}$$

Consequently the projection operator $\hat{r}(\hat{\mathbf{n}})_\alpha \equiv \hat{r}(\theta, \varphi)_\alpha$ of the position of the αth particle along $\hat{\mathbf{n}}$ coincides with the coordinate operator $\hat{r}(g_0)_{\alpha,3}$:

$$\hat{r}(\hat{\mathbf{n}})_\alpha \equiv \hat{r}(\theta, \varphi)_\alpha = \hat{r}(g_0)_{\alpha,3}. \tag{6.52}$$

The last results together with (6.49) allows us to compute the dispersion of $\hat{r}(\theta, \varphi)_\alpha$ for the αth particle in the direction $\hat{\mathbf{n}}(\theta, \varphi)$ whenever the system is in the state $|p; n\rangle$:

$$\begin{aligned} \mathrm{Disp}((\hat{r}(\theta, \varphi))_\alpha)_{|p;n\rangle} &= \mathrm{Disp}(\hat{r}(g_0)_{\alpha,3})_{|p;n\rangle} = (n_1 + n_{\alpha+3}) cos^2\varphi\ sin^2\theta \\ &+ (n_2 + n_{\alpha+3}) sin^2\varphi\ sin^2\theta + (n_3 + n_{\alpha+3}) cos^2\theta. \end{aligned} \tag{6.53}$$

From the inequality $n_k + n_{\alpha+3} \le p$, we deduce

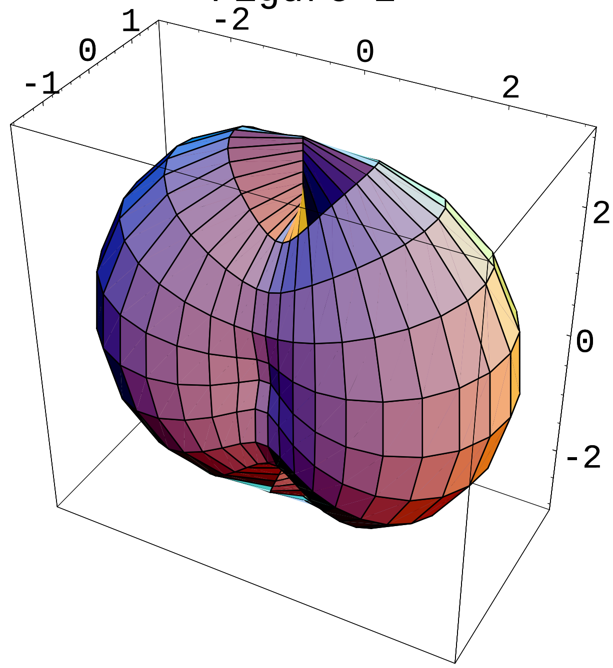
$$\mathrm{Disp}((\hat{r}(\theta, \varphi))_\alpha)_{|p;n\rangle} \le p. \tag{6.54}$$

The last expression holds for any basis vector $|p; n\rangle$ from the state space $W(N, p)$, for any direction $\hat{\mathbf{n}}(\theta, \varphi)$ and for any particle α. Skipping these labels we can write $\mathrm{Disp}(\hat{r}) \le p$. Taking into account that $\hat{p}^2_{\alpha,i} = \hat{r}^2_{\alpha,i}$ one also concludes that $\mathrm{Disp}(\hat{p}) \le p$. Thus, for the standard deviations we finally obtain:

$$\Delta\hat{r} \le \sqrt{p}, \quad \Delta\hat{p} \le \sqrt{p}. \tag{6.55}$$

On Figure 2 we have given the standard deviation of the first particle whenever the oscillator is in the state $|p; 3, 0, 8; 1, n_5, ..., n_{N+3}\rangle$. Observe that the standard deviation is within a parallelepiped, the length of each edge of which is less than $\sqrt{1}2$ since the minimal possible value of p is 12.

Figure 2



The conclusion is that the uncertainty of the position and of the momentum along any direction, for any particle and for any state is less than $\sqrt{p}$. Consequently for the analogue of the uncertainty relations in QM we have

$$\Delta\hat{r}\Delta\hat{p} \leq p. \tag{6.56}$$

Then for the initial position and momentum operators, see (6.9), eq.(6.55) yields:

$$\Delta\hat{R}\Delta\hat{P} \leq \frac{p\hbar}{|N-3|}. \tag{6.57}$$

The above equation is very different from the corresponding uncertainty relations in QM, for instance $\Delta x \Delta p_x \geq \frac{\hbar}{2}$. In particular the increasing of the accuracy of the position of a particle does not force the uncertainty of the momentum of the same particle to increase.

We proceed to evaluate the dispersion of $\hat{r}(g)^2_{\alpha,k}$ for any g in the state $|p;n\rangle$. For this purpose it suffices to compute the transformation of the basis $|p;n\rangle$ under the action of $\hat{r}(g)^2_{\alpha,k}$ for any g and $k=1,2,3$. Here is the result:

$$\begin{aligned}
\hat{r}(g)^2_{\alpha,k}|p;n\rangle = & \big(g^2_{1k}(n_1+n_{\alpha+3}) + g^2_{2k}(n_2+n_{\alpha+3}) + g^2_{3k}(n_3+n_{\alpha+3})\big)|p;n\rangle \\
& + g_{1k}g_{2k}(\sqrt{(n_1+1)n_2}|p;n\rangle_{1,-2} + \sqrt{(n_2+1)n_1}|p;n\rangle)_{-1,2}) \\
& + g_{1k}g_{3k}(\sqrt{(n_1+1)n_3}|p;n\rangle_{1,-3} + \sqrt{(n_3+1)n_1}|p;n\rangle_{-1,3}) \\
& + g_{2k}g_{3k}(\sqrt{(n_3+1)n_2}|p;n\rangle_{-2,3} + \sqrt{(n_2+1)n_3}|p;n\rangle_{2,-3})
\end{aligned} \tag{6.58}$$

This result is in agreement with (6.48), since $(|p;n\rangle, \hat{r}(g)^2_{\alpha,k}|p;n\rangle)$ gives the same expression for $\langle \hat{r}(g)^2_{\alpha,k}\rangle_{|p;n\rangle}$. Moreover (6.58) helps to compute $\langle \hat{r}(g)^4_{\alpha,k}\rangle_{|p;n\rangle}$ using the hermiticity of $\hat{r}(g)_{\alpha,k}$:

$$\begin{aligned}\langle \hat{r}(g)^4_{\alpha,k}\rangle_{|p;n\rangle} &= (|p;n\rangle, \hat{r}(g)^4_{\alpha,k}|p;n\rangle) = (\hat{r}(g)^2_{\alpha,k}|p;n\rangle, \hat{r}(g)^2_{\alpha,k}|p;n\rangle)\\ &= \big(g^2_{1k}(n_1+n_{\alpha+3}) + g^2_{2k}(n_2+n_{\alpha+3}) + g^2_{3k}(n_3+n_{\alpha+3})\big)^2\\ &+ g^2_{1k}g^2_{2k}(2n_1n_2+n_1+n_2) + g^2_{1k}g^2_{3k}(2n_1n_3+n_1+n_3)\\ &+ g^2_{3k}g^2_{2k}(2n_2n_3+n_2+n_3).\end{aligned} \tag{6.59}$$

Hence for the dispersion of $(\hat{r}(g)_{\alpha,k})^2 \equiv \hat{r}(g)^2_{\alpha,k}$ in the state $|p;n\rangle$ we finally obtain

$$\begin{aligned}&\mathrm{Disp}(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle} = \langle \hat{r}(g)^4_{\alpha,k}\rangle_{|p;n\rangle} - \langle \hat{r}(g)^2_{\alpha,k}\rangle^2_{|p;n\rangle}\\ &= g^2_{1k}g^2_{2k}(2n_1n_2+n_1+n_2) + g^2_{1k}g^2_{3k}(2n_1n_3+n_1+n_3)\\ &+ g^2_{3k}g^2_{2k}(2n_2n_3+n_2+n_3), \quad k=1,2,3, \quad \alpha=1,...,N.\end{aligned} \tag{6.60}$$

which can be written also as

$$\mathrm{Disp}(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle} = \sum_{i<j=1}^{3} g^2_{ik}g^2_{jk}(2n_in_j+n_i+n_j). \tag{6.61}$$

Then the expression for the standard deviation of $\hat{r}(g)^2_{\alpha,k}$ in the state $|p;n\rangle$ read:

$$\begin{aligned}&\Delta(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle} = \Big(g^2_{1k}g^2_{2k}(2n_1n_2+n_1+n_2)\\ &+ g^2_{1k}g^2_{3k}(2n_1n_3+n_1+n_3) + g^2_{3k}g^2_{2k}(2n_2n_3+n_2+n_3)\Big)^{1/2}.\end{aligned} \tag{6.62}$$

The problem to solve now is to find all different reference frames $\mathbf{e}(g)$ for which the dispersion (6.61) vanishes simultaneously for all three values of $k=1,2,3$. If $\mathbf{e}(\bar{g}) \equiv (\mathbf{e}(\bar{g})_1, \mathbf{e}(\bar{g})_2, \mathbf{e}(\bar{g})_3)$ is one such frame, then according to Conclusion 6.2 the state $|p;n\rangle$ will be a common eigenstate of $\hat{r}(\bar{g})^2_{\alpha,1}$, $\hat{r}(\bar{g})^2_{\alpha,2}$ and $\hat{r}(\bar{g})^2_{\alpha,3}$. This means that the nondiagonal terms in (6.58) have to vanish so that:

$$\hat{r}(\bar{g})^2_{\alpha,k}|p;n\rangle = \big((\bar{g}^2_{1k}(n_1+n_{\alpha+3}) + \bar{g}^2_{2k}(n_2+n_{\alpha+3}) + \bar{g}^2_{3k}(n_3+n_{\alpha+3})\big)|p;n\rangle, \tag{6.63}$$

The eigenvalues

$$r(\bar{g})^2_{\alpha,k} = \bar{g}^2_{1k}(n_1+n_{\alpha+3}) + \bar{g}^2_{2k}(n_2+n_{\alpha+3}) + g^2_{3k}(n_3+n_{\alpha+3}), \tag{6.64}$$

of $\hat{r}(\bar{g})^2_{\alpha,k}$, $k = 1, 2, 3$, are the squares of the admissible coordinates of the αth particle in the frame $\mathbf{e}(\bar{g})$. Let us summarize.

Conclusion 6.2. *Let the system be in the state $|p; n\rangle$. If the dispersion* $\mathrm{Disp}(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle}$ *vanishes for a certain g and for all $k = 1, 2, 3$, i.e.,*

$$\begin{aligned}\mathrm{Disp}(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle} &= g^2_{1k}g^2_{2k}(2n_1n_2 + n_1 + n_2)\\ &+ g^2_{1k}g^2_{3k}(2n_1n_3 + n_1 + n_3) + g^2_{3k}g^2_{2k}(2n_2n_3 + n_2 + n_3) = 0,\end{aligned} \tag{6.65}$$

then $|p; n\rangle$ is an eigenstate of $\hat{r}(g)^2_{\alpha,k}$,

$$\hat{r}(g)^2_{\alpha,k}|p; n\rangle = r(g)^2_{\alpha,k}|p; n\rangle, \tag{6.66}$$

with eigenvalues

$$r(g)^2_{\alpha,k} = g^2_{1k}(n_1 + n_{\alpha+3}) + g^2_{2k}(n_2 + n_{\alpha+3}) + g^2_{3k}(n_3 + n_{\alpha+3}), \quad k = 1, 2, 3. \tag{6.67}$$

In such a case

$$\Gamma(|p; n\rangle), \alpha, g) = \{r(g)_{\alpha,1}\mathbf{e}(g)_1 + r(g)_{\alpha,2}\mathbf{e}(g)_2 + r(g)_{\alpha,3}\mathbf{e}(g)_3\} \tag{6.68}$$

determines admissible places, i.e., nests for the αth particle.

The validity of the above proposition can be verified also directly. To this end write the dispersion (6.60) as follows:

$$\begin{aligned}D(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle} =& g^2_{1k}g^2_{2k}(n_1 + 1)n_2 + g^2_{1k}g^2_{2k}(n_2 + 1)n_1\\ &+ g^2_{1k}g^2_{3k}(n_1 + 1)n_3 + g^2_{1k}g^2_{3k}(n_3 + 1)n_1\\ &+ g^2_{2k}g^2_{3k}(n_2 + 1)n_3 + g^2_{2k}g^2_{3k}(n_3 + 1)n_2.\end{aligned} \tag{6.69}$$

Since the dispersion (6.69) is a sum of nonnegative terms, it vanishes only if every term vanishes, i.e. if for any $i < j = 1, 2, 3$,

$$g^2_{ik}g^2_{jk}(n_i + 1)n_j = 0 \ \text{ and } \ g^2_{ik}g^2_{jk}(n_j + 1)n_i = 0.$$

But then

$$g_{ik}g_{jk}\sqrt{(n_i + 1)n_j} = 0 \ \text{ and } \ g_{ik}g_{jk}\sqrt{(n_j + 1)n_i} = 0, \quad i < j = 1, 2, 3$$

and therefore all off diagonal terms in (6.58) vanish. Hence (6.66) and (6.67) hold.

The conclusion 5.7 is clear. What is not so clear is how many if any are the new nests in addition to (6.24). We shall answer this question first on the example, namely one-particle 3D oscillator with $p = 1$.

Example 6.2. Consider one-particle Wigner oscillator in a representation $p = 1$. The state space $V(N = 1, p = 1)$ is 4-dimensional with a basis

$$\varphi_1 = |p = 1; 1, 0, 0, 0\rangle,\ \varphi_2 = |p = 1; 0, 1, 0, 0\rangle,\ \varphi_3 = |p = 1; 0, 0, 1, 0\rangle,\ \varphi_4 = |p = 1; 0, 0, 0, 1\rangle. \tag{6.70}$$

1a. Take first $\varphi_3 = |p = 1; 0, 0, 1; 0\rangle$. The requirement the dispersion $\mathrm{Disp}(\hat{r}(g)_k^2)_{\varphi_3}$, $g = g(\alpha, \beta, \gamma)$, to vanish reads, see (6.60):

$$\mathrm{Disp}(\hat{r}(g)_k^2)_{\varphi_3} = g_{1k}^2 g_{3k}^2 + g_{2k}^2 g_{3k}^2 = 0, \quad k = 1, 2, 3. \tag{6.71}$$

Since the system consists of only one particle, we have suppressed in (6.71) the index α.

The simplest solution of (6.71) corresponds to $\alpha = \beta = \gamma = 0$, namely to g being a 3×3 unit matrix, $g = 1$. In this case

$$\hat{r}_1^2 \varphi_3 = 0, \quad \hat{r}_2^2 \varphi_3 = 0, \quad \hat{r}_3^2 \varphi_3 = \varphi_3. \tag{6.72}$$

Then according to Conclusion 5.7

$$\Gamma(|p = 1; 0, 0, 1; 0\rangle, \alpha = 1, g = 1) = \{\pm \mathbf{e}_3\} \equiv (0, 0, \pm 1), \tag{6.73}$$

i.e., there are two nests A with coordinates $(0, 0, 1)$ and B with coordinates (0,0,-1) where the particle can be registered to be (see Figure 3).

Apart from $g = 1$, the Eqs. (6.71) have also other solutions. It takes some time to find all of them (see corollary B.1 for more details):

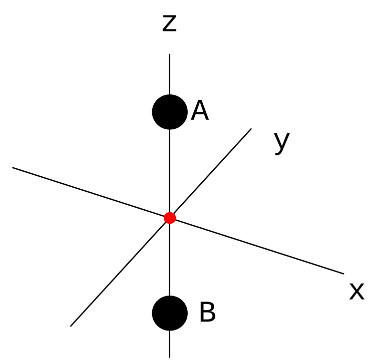
$$a) \quad \beta \in \{0,\ \pi\},\ \alpha, \gamma \ - \text{arbitrary}, \tag{6.74a}$$

$$b) \quad \beta = \pi/2,\ \gamma \in \{0,\ \pi/2,\ \pi,\ 3\pi/2\}, \quad \alpha - \text{arbitrary}, \tag{6.74b}$$

Do they lead to new nests? No, they do not. It turns out that any of the above solutions corresponds to the same picture shown on Figure 1. Let us illustrate this on an example with $\beta = \pi/2$, $\gamma = 0$ and α arbitrary. In this case

$$g_3 \equiv g(\alpha, \beta = \pi/2, \gamma = 0) = \begin{pmatrix} 0, & -\sin\alpha, & \cos\alpha \\ 0, & \cos\alpha, & \sin\alpha \\ 1, & 0 & 0 \end{pmatrix}, \tag{6.74}$$

Figure 3



and (6.58) yields:

$$\hat{r}(g)_1^2\varphi_3 = \varphi_3,\ \hat{r}(g)_2^2\varphi_3 = 0,\ \hat{r}(g)_3^2\varphi_3 = 0. \tag{6.76}$$

Therefore

$$r(g)_1^2 = 1,\ r(g)_2^2 = 0,\ r(g)_3^2 = 0, \tag{6.77}$$

so for the nests we obtain applying (6.68)

$$\Gamma(|p = 1; 0, 0, 1; 0\rangle, \alpha = 1, g_3) = \{\pm\mathbf{e}(g)_1\}. \tag{6.78}$$

But $\mathbf{e}(g)_1 = \sum_j \mathbf{e}_j g_{j1} = \mathbf{e}_3$, i.e., we obtain the same nests as in Figure 1:

$$\Gamma(|p = 1; 0, 0, 1; 0\rangle, \alpha = 1, g = 1) = \Gamma(|p = 1; 0, 0, 1; 0\rangle, \alpha = 1, g_3) = \{\pm\mathbf{e}_3\}, \tag{6.79}$$

The proof for the other cases in (6.74) is similar and it gives all of the time the nests as shown on Figure 1.

1b. In an analogues way one finds that the space configuration of the state $|p = 1; 1, 0, 0, 0\rangle$ (resp. $|p = 1; 0, 1, 0, 0\rangle$,) corresponds to a picture with two nests $\pm\mathbf{e}_1$ (resp. $\pm\mathbf{e}_2$).

1c. The above results suggest that the nests $\Gamma(\varphi_k)$, corresponding to $g = 1$, see (6.73), contain already all nests where the particle can be accommodated. Is this the case?

No, it is not. In order to show this we consider the last basis state φ_4, corresponding to $n_1 = n_2 = n_3 = 0$ and $n_4 = 1$.

Evidently, see (6.60), the dispersion $\mathrm{Disp}(\hat{r}(g)^2_{\alpha,k})_{\varphi_4} = 0$ for any g. Then, as it should be, φ_4 is an eigenstate of $\hat{r}(g)^2_k$ for any g. Indeed, setting in (6.58) $n_1 = n_2 = n_3 = 0$, one has

$$\hat{r}(g)^2_k \varphi_4 = (g^2_{1k} + g^2_{2k} + g^2_{3k})\varphi_4. \tag{6.80}$$

The matrix g is an orthogonal matrix, $gg^t = 1$, and therefore $g^2_{1k} + g^2_{2k} + g^2_{3k} = 1$. Thus, for any g,

$$\hat{r}(g)^2_k \varphi_4 = \varphi_4, \quad k = 1, 2, 3.$$

Since $\hat{r}(g)^2_k$ is square coordinate operator along $\mathbf{e}(g)_k$ and the basis $\mathbf{e}(g)$ is orthonormed, the conclusion is that any point on a sphere with radius $\sqrt{3}$ is a nest.

We see that the properties of the state φ_4 are very different from the properties of the other three basis vectors. In particular if the system is in this state, the particle can be observed in every point of the sphere with radius $\sqrt{3}$.

Passing to the general case we divide all states into three nonintersecting classes.

Class I. All states $|p;n\rangle$ with $n_1 = n_2 = n_3 = 0$;

Class II. All states $|p;n\rangle$ for which two of the integers n_1, n_2, n_3 do not vanish.

Class III. All states $|p;n\rangle$ for which two and only two of the integers n_1, n_2, n_3 vanish.

1. Properties of the states from Class I

Observe first of all that the Class I is not empty only if $p \leq N$. The latter stems from the observation that the sum of the fermionic coordinates n_f of any state $|p;n\rangle$ cannot exceed N, whereas $n_1 + ... + n_{N+3} = p$. Therefore if $p > N$, then at least one of the bosonic coordinates n_1, n_2, n_3 of $|p;n\rangle$ cannot vanish.

1a. The next property is almost evident from the results just proved (see part 1c in the example 6.2).

Corollary 6.3. *If the system is in a state $|p;0,0,0;...,1_{\alpha+3},..\rangle$ from Class I, then any point on a sphere with radius $\sqrt{3}$ is a nest for $\alpha$the particle.*

1b. The state $\psi = |p; 0, 0, 0; ..., n_{\alpha+3} = 0, ..\rangle$ from Class I is of particular interest. One verifies that

$$\hat{H}_\alpha \psi = \hat{S}_{\alpha k} \psi = \hat{R}_{\alpha k} \psi = \hat{P}_{\alpha k} \psi = 0, \quad k = 1, 2, 3. \tag{6.81}$$

Hence,

Corollary 6.4. *A state*

$$|p; 0, 0, 0; n_4, ..., n_{\alpha+2}, 0_{\alpha+3}, n_{\alpha+4}, ..., n_{N+3}\rangle \tag{6.82}$$

corresponds to a space configuration of the system when the αth particle "condensates" onto the origin of the oscillating system with zero energy, zero momentum and zero angular momentum.

The property that some of the oscillating particles can condensate onto the origin with zero energy exhibits another difference with the conventional case: in canonical QM the ground energy of any $3D$ free harmonic oscillator is never zero, it cannot be less $\frac{3}{2}\omega\hbar$.

2. Properties of the states from Class II

We have already indicated in proposition 6.4 that if the system is in the state $|p, n\rangle$ then the collection of points

$$\Gamma(|p; n\rangle)_\alpha = \{\pm\sqrt{n_1 + n_{\alpha+3}}\ \mathbf{e}_1 \pm \sqrt{n_2 + n_{\alpha+3}}\ \mathbf{e}_2 \pm \sqrt{n_3 + n_{\alpha+3}}\ \mathbf{e}_3\}| \tag{6.83}$$

are nests for αth particle. Now we prove a stronger statement.

Proposition 6.6. *If the system is in a basis state $|p; n\rangle$ from Class II, then the nests (6.83) are the only nests for the αth particle.*

For the proof of this relatively long proposition see appendix B.

3. Properties of the states from Class III

It remains to investigate the space structure of the basis states from the Class III, namely all those basis states for which two and only two of the bosonic coordinates n_1, n_2, n_3 of $|p; n\rangle$ vanish.

Let us consider for definiteness a state $|p; 0, 0, n_3, .., n_{\alpha+3}, ...\rangle$. In this case, see (6.60), the condition the dispersion of $\hat{r}(g)^2_{\alpha k}$ to vanish reduces to

$$(g^2_{1k}g^2_{3k} + g^2_{2k}g^2_{3k})n_3 = 0, \tag{6.84}$$

and since $n_3 \neq 0$, (6.84) is equivalent to

$$g^2_{1k}g^2_{3k} = 0, \quad g^2_{2k}g^2_{3k} = 0. \tag{6.85}$$

We have already found all solutions of Eqs. (6.85), see corollary B.1 (in appendix B). If the $g-$matrix is one such solution, then the state $|p; 0, 0, n_3, ..., n_{\alpha+3}, ..\rangle$ is an eigenvector of $\hat{r}(g)^2_{\alpha k}$,

$$\hat{r}(g)^2_{\alpha k}|p; 0, 0, n_3, ..., n_{\alpha+3}, ..\rangle = r(g)^2_{\alpha k}|p; 0, 0, n_3, ..., n_{\alpha+3}, ..\rangle \tag{6.86}$$

with

$$r(g)^2_{\alpha k} = g^2_{3,k}n_3 + n_{\alpha+3}. \tag{6.87}$$

Assume the system is in the state $|p; 0, 0, n_3, ..., n_{\alpha+3}, ..\rangle$ from Class III. Also in this case the results depend essentially on the value of $n_{\alpha+3}$.

3a. Let $n_{\alpha+3} = 0$. Then the αth particle has only two nests, namely $\pm\sqrt{n_3}\mathbf{e}_3$. The proof of this result is essentially the same as in example 6.2, part 1c, so we omit it. In a similar way one establishes the space structure of the αth particle in the states $|p; 0, n_2, 0, ..., 0_{\alpha+3}, ..\rangle$ and $|p; n_1, 0, 0, ..., 0_{\alpha+3}, ..\rangle$. The results are collected in the next proposition.

Proposition 6.7. *The correspondence state - space structure for the αth particle read:*

$$|p; 0, 0, n_3, ..., 0_{\alpha+3}, ..\rangle \quad \leftrightarrow \quad \pm\sqrt{n_3}\mathbf{e}_3, \quad n_3 \neq 0, \tag{6.88a}$$

$$|p; 0, n_2, 0, ..., 0_{\alpha+3}, ..\rangle \quad \leftrightarrow \quad \pm\sqrt{n_2}\mathbf{e}_2, \quad n_2 \neq 0, \tag{6.88b}$$

$$|p; n_1, 0, 0, ..., 0_{\alpha+3}, ..\rangle \quad \leftrightarrow \quad \pm\sqrt{n_1}\mathbf{e}_1, \quad n_1 \neq 0, \tag{6.88c}$$

3b. The case with $n_{\alpha+3} = 1$ is more involved. Here is the result. For the proof see Appendix C.

Proposition 6.8

1. Whenever the system is in a state $|p; n_1, 0, 0, .., 1_{\alpha+3}, ..\rangle$ all nests of the αth particle are

$$\begin{aligned}\Gamma\big(|p; n_1, 0, 0, ..., 1_{\alpha+3}, ..\rangle,\ \alpha\big) = \big\{\xi_1\sqrt{n_1+1}\ \mathbf{e}_1 + (\cos\alpha - \sin\alpha)\mathbf{e}_2 \\ + (\sin\alpha + \cos\alpha)\mathbf{e}_3 \mid \alpha \in \mathbf{R}, \quad \xi_1 = \pm 1, \big\}, \quad n_1 \neq 0.\end{aligned} \tag{6.89}$$

2. *All nests of the αth particle whenever the system is in a state $|p;0,n_2,0,..,1_{\alpha+3},..\rangle$ are*

$$\Gamma\big(|p;0,n_2,0,...,1_{\alpha+3},..\rangle,\ \alpha\big)=\big\{(\cos\alpha-\sin\alpha)\mathbf{e}_1+\xi_2\sqrt{n_2+1}\ \mathbf{e}_2 \\ +(\sin\alpha+\cos\alpha)\mathbf{e}_3 \mid \alpha\in\mathbf{R},\quad \xi_2=\pm1,\big\},\quad n_2\neq 0. \tag{6.90}$$

3. *All nests of the αth particle whenever the system is in a state $|p;0,0,n_3,..,1_{\alpha+3},..\rangle$ are*

$$\Gamma\big(|p;0,0,n_3,...,1_{\alpha+3},..\rangle,\ \alpha\big)=\big\{(\cos\alpha-\sin\alpha)\mathbf{e}_1 \\ +(\sin\alpha+\cos\alpha)\mathbf{e}_2+\xi_3\sqrt{n_3+1}\ \mathbf{e}_3 \mid \alpha\in\mathbf{R},\quad \xi_3=\pm1,\big\},\quad n_3\neq 0. \tag{6.91}$$

From the results obtained so far we can draw already some conclusions about the collective properties of the system. First we observe that the dispersion (6.60) for the αth particle does not depend on α. This leads to the following conclusion:

Corollary 6.5. *Let the system be in an arbitrary basis state $|p;n\rangle$. Then if the dispersion $\mathrm{Disp}(\hat{\mathrm{r}}(\mathrm{g})^2_{\alpha_0,\mathrm{k}})_{|\mathrm{p;n}\rangle}$, $\mathrm{k}=1,2,3$, see (6.60), vanishes for one particular particle α_0, then it vanishes for all particles. Therefore if $|p;n\rangle$ is an eigenstate of $\hat{r}(g)^2_{\alpha_0,1}$, $\hat{r}(g)^2_{\alpha_0,2}$, $\hat{r}(g)^2_{\alpha_0,3}$ for one particular particle α_0, then it is an eigenstate of $\hat{r}(g)^2_{\alpha,1}$, $\hat{r}(g)^2_{\alpha,2}$, $\hat{r}(g)^2_{\alpha,3}$ for all particles $\alpha=1,2,...,N$. Consequently, all observables $\hat{r}(g)^2_{\alpha,1}$, $\hat{r}(g)^2_{\alpha,2}$, $\hat{r}(g)^2_{\alpha,3}$, $\alpha=1,2,...,N$ can be measured simultaneously whenever the system is in the state $|p;n\rangle$.*

Observe next that whenever the system is in a state $|p;n\rangle$ the coordinates of the nests of a particle $\#\alpha$ do not depend explicitly on α. The dependence is indirect, via $n_{\alpha+3}$, which can take only two values, 0 or 1. As a consequence one concludes:

Corollary 6.6. *Given a basic state $|p;n\rangle$. All particles $\alpha_1,\alpha_2,...,\alpha_k$ for which $n_{\alpha_1+3}=n_{\alpha_2+3}=...=n_{\alpha_k+3}=0$ have common nests; the nests of the rest of the particles, namely those for which $n_{\alpha_{k+1}}=n_{\alpha_{k+2}}=...=n_{\alpha_{N+3}}=1$ also coincide.*

Consider for instance a $6-$particle oscillator in a state $\varphi_1=|p=4;0,0,1;0,0,0,1,1,1\rangle$ (from the Class III). Then according to proposition 6.7 the first three particles $\#1,2,3$ have two common nests: $\mathbf{e}_3$ and $-\mathbf{e}_3$. The space configuration of particles $4,5,6,$ is very

different. Similar as on Figure 3, each particle is accommodated somewhere on two circles with radius $\sqrt{2}$ around $z-$axes, which are on a distance $\sqrt{2}$ above or below the $x0y$.

7. Physical properties - angular momentum, parity and probability distributions

Here we discuss as a first step the properties of the angular momentum of the oscillator system. The results are very different from the angular momentum properties of both the conventional oscillators and the WQSs studied so far. The most unusual new feature is that the operators of the projections of the angular momentum are the same for all particles, see (3.28), and they coincide with the generators of the algebra $so(3)$ of the rotation group. As a second step we introduce another important physical observable, namely the parity P of the states and finally we use it in order to show that any particle occupies with equal probability anyone of its nests.

Explicitly the physical $so(3)$ generators, defined as operators in $W(3|N)$ read:

$$\hat{S}_1 = \mathrm{i}(b_3^+ b_2^- - b_2^+ b_3^-), \quad \hat{S}_2 = \mathrm{i}(b_1^+ b_3^- - b_3^+ b_1^-), \quad \hat{S}_3 = \mathrm{i}(b_2^+ b_1^- - b_1^+ b_2^-). \tag{7.1}$$

Then the angular momentum projections of the αth particle are

$$\hat{M}_{\alpha i} = \frac{\hbar}{N-3}\hat{S}_i, \quad i = 1, 2, 3, \tag{7.2}$$

whereas for the components of the angular momentum of the entire oscillator one has

$$\hat{M}_j = \frac{\hbar N}{N-3}\hat{S}_j. \tag{7.3}$$

As a result the oscillating particles behave as if they were charged particles in a strong magnetic field: the angular momentums of all particles are parallel to each other.

The operators (7.1) are not diagonal in the basis (4.2). We proceed to introduce a new basis which diagonalizes $\hat{S}_3$. To this end consider the unitary matrix

$$(G) \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-\mathrm{i}}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_N \end{pmatrix}. \tag{7.4}$$

where 1_N is an $N-$dimensional unit matrix. Let $c^\pm = (c_1^\pm, c_2^\pm, \ldots, c_{N+3}^\pm)$. Denote by G^+ the Hermitian conjugate to G matrix. Then the operators

$$c(G)_i^+ = (c^+ G)_i, \quad c(G)_i^- = (G^+ c^-)_i, \quad i = 1, 2, .., N+3, \tag{7.5}$$

satisfy the conditions listed in (4.1). More explicitly,

1. $$c(G)_1^+ \equiv B_1^+ = \frac{1}{\sqrt{2}}(b_1^+ - \mathrm{i}b_2^+),\; c(G)_2^+ \equiv B_2^+ = \frac{1}{\sqrt{2}}(b_1^+ + \mathrm{i}b_2^+),\; c(G)_3^+ \equiv B_3^+ = b_3^+, \quad (7.6)$$

$$c(G)_1^- \equiv B_1^- = \frac{1}{\sqrt{2}}(b_1^- + \mathrm{i}b_2^-),\quad c(G)_2^+ \equiv B_2^- = \frac{1}{\sqrt{2}}(b_1^- - \mathrm{i}b_2^-),\quad c(G)_3^- \equiv B_3^- = b_3^-, \quad (7.7)$$

are Bose operators and odd elements (as linear combination of odd elements),

2. $$c(G)_A^\pm \equiv F_A^\pm = f_A^\pm, \quad A = 4, 5, ..., N+3, \qquad (7.8)$$

are Fermi operators and even elements.

3. The Bose operators anticommute with Fermi operators.

An immediate consequence of the above results is that all vectors

$$\begin{aligned} |p; n_1, n_2, \ldots, n_{N+3}) = & \frac{(c(G)_1^+)^{n_1}(c(G)_2^+)^{n_2}\ldots(c(G)_{N+3}^+)^{n_{N+3}}}{\sqrt{n_1!n_2!n_3!}}|0\rangle, \\ = & \frac{(B_1^+)^{n_1}(B_2^+)^{n_2}(b_3^+)^{n_3}(f_4^+)^{n_4}\ldots(f_{N+3}^+)^{n_{N+3}}}{\sqrt{n_1!n_2!n_3!}}|0\rangle, \end{aligned} \qquad (7.9)$$

where

$$n_1, n_2, n_3 \in \mathbf{Z}_+, \quad n_4, n_5, \ldots, n_{N+3} \in \{0, 1\}, \quad \text{and} \quad n_1 + ... + n_{N+3} = p, \qquad (7.10)$$

constitute an orthonormed basis in $W(3|N)$. The transformation of the new basis under the action of the CAOs $B_1^\pm, B_2^\pm, B_3^\pm$ and the Fermi CAOs is the same as in (4.4) with $b_i^\pm$ replaced by $B_i^\pm$, i.e.

$$B_i^+|.., n_i, ..) = \sqrt{n_i + 1}|.., n_i + 1, ..), \quad i = 1, 2, 3; \qquad (7.11a)$$

$$B_i^-|.., n_i, ..) = \sqrt{n_i}|.., n_i - 1, ..), \quad i = 1, 2, 3; \qquad (7.11b)$$

$$f_i^+|.., n_i, ..) = (-1)^{n_1+..+n_{i-1}}\sqrt{1 - n_i}|.., n_i + 1, ..), \quad i = 4, 5, .., N + 3; \qquad (7.11c)$$

$$f_i^-|.., n_i, ..) = (-1)^{n_1+..+n_{i-1}}\sqrt{n_i}|.., n_i - 1, ..), \quad i = 4, 5, .., N + 3; \qquad (7.11d)$$

Taking into account that

$$b_1^+ = \frac{1}{\sqrt{2}}(B_1^+ + B_2^+), \quad b_2^+ = \frac{\mathrm{i}}{\sqrt{2}}(B_1^+ - B_2^+), \quad b_3^+ = B_3^+, \tag{7.12a}$$

$$b_1^- = \frac{1}{\sqrt{2}}(B_1^- + B_2^-), \quad b_2^- = \frac{-\mathrm{i}}{\sqrt{2}}(B_1^- - B_2^-), \quad b_3^- = B_3^-, \tag{7.12b}$$

we obtain from (7.1)

$$\hat{S}_1 = \mathrm{i}(b_3^+ b_2^- - b_2^+ b_3^-) = \frac{1}{\sqrt{2}}(B_3^+ B_1^- - B_3^+ B_2^- + B_1^+ B_3^- - B_2^+ B_3^-), \tag{7.13a}$$

$$\hat{S}_2 = \mathrm{i}(b_1^+ b_3^- - b_3^+ b_1^-) = \frac{\mathrm{i}}{\sqrt{2}}(-B_3^+ B_1^- - B_3^+ B_2^- + B_1^+ B_3^- + B_2^+ B_3^-). \tag{7.13b}$$

$$\hat{S}_3 = \mathrm{i}(b_2^+ b_1^- - b_1^+ b_2^-) = B_2^+ B_2^- - B_1^+ B_1^-. \tag{7.13c}$$

Then

$$\hat{S}_+ = \sqrt{2}(B_3^+ B_1^- - B_2^+ B_3^-), \quad \hat{S}_- = \sqrt{2}(B_1^+ B_3^- - B_3^+ B_2^-), \tag{7.14}$$

and

$$\begin{aligned} \hat{S}_+|p; n_1, n_2, n_3, \ldots, n_{N+3}) &= \sqrt{2(n_3+1)n_1}|p; n_1 - 1, n_2, n_3 + 1, \ldots, n_{N+3}) \\ &- \sqrt{2(n_2+1)n_3}|p; n_1, n_2 + 1, n_3 - 1, \ldots, n_{N+3}), \quad (7.15a) \\ \hat{S}_-|p; n_1, n_2, n_3, \ldots, n_{N+3}) &= \sqrt{2(n_1+1)n_3}|p; n_1 + 1, n_2, n_3 - 1, \ldots, n_{N+3}) \\ &- \sqrt{2(n_3+1)n_2}|p; n_1, n_2 - 1, n_3 + 1, \ldots, n_{N+3}), \quad (7.15b) \end{aligned}$$

The operator $\hat{S}_3$ is diagonal in the basis $|n_1, n_2, \ldots, n_{N+3})$:

$$\hat{S}_3|p; n_1, n_2, \ldots, n_{N+3}) = (n_2 - n_1)|p; n_1, n_2, \ldots, n_{N+3}). \tag{7.16}$$

Hence $\hat{S}_3 = \mathrm{i}(b_2^+ b_1^- - b_1^+ b_2^-)$ is diagonal in the basis (7.9), but written in terms of the initial CAOs (7.12) namely

$$|p; n_1, n_2, \ldots, n_{N+3}) = \frac{(\frac{1}{\sqrt{2}}(b_1^+ - \mathrm{i}b_2^+))^{n_1}(\frac{1}{\sqrt{2}}(b_1^+ + \mathrm{i}b_2^+))^{n_2}(b_3^+)^{n_3}(f_4^+)^{n_4}\ldots(f_{N+3}^+)^{n_{N+3}}}{\sqrt{n_1!n_2!n_3!}}|0\rangle \tag{7.17}$$

We call the basis (7.9) (resp. 6.17) S_3-*basis* and abbreviate

$$|p; n_1, ..., n_{N+3}) \equiv |p; n). \tag{7.18}$$

For the $so(3)$ Casimir operator

$$\hat{\mathbf{S}}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 = \hat{S}_+\hat{S}_- + \hat{S}_3^2 - \hat{S}_3 \tag{7.19}$$

we find

$$\begin{aligned}\hat{\mathbf{S}}^2|p;n_1,..,n_{N+3}) =& \big(2(n_1+1)n_3 + 2(n_3+1)n_2 + (n_2-n_1)^2 - n_2 + n_1\big)|p;n_1,..,n_{N+3})\\ &-2\sqrt{(n_1+1)(n_2+1)n_3(n_3-1)}|p;n_1+1,n_2+1,n_3-2,n_4,..,n_{N+3})\\ &2\sqrt{(n_3+1)(n_3+2)n_1n_2}|p;n_1-1,n_2-1,n_3+2,n_4,..,n_{N+3})\end{aligned}$$

This result is not unexpected. The $so(3)$ Casimir operator is not proportional to the unity because the corresponding representation is not irreducible (and is not a direct sum of irreps with the same signature). For further use we formulate a

Proposition 7.1. All vectors $|p;n_1,n_2,n_3,n_4,...,n_{N+3})$ with one and the same $n_b = n_1+n_2+n_3$ and fixed $n_4,...,n_{N+3}$ are vectors from $V(N,p,n_b,n_4,...,n_{N+3})$ and therefore they have one and the same energy (5.4).

Indeed,

$$\begin{aligned}(B_1^+)^{n_1}(B_2^+)^{n_2} &= \Big(\frac{b_1^+ - \mathrm{i}b_2^+}{\sqrt{2}}\Big)^{n_1}\Big(\frac{b_1^+ + \mathrm{i}b_2^+}{\sqrt{2}}\Big)^{n_2}\\ &= \sum_{k=0}^{n_1}\sum_{q=0}^{n_2} c(n_1,n_2,k,q)(b_1^+)^{n_1+n_2-k-q}(b_2^+)^{k+q},\end{aligned}$$

where $c(n_1,n_2,k,q)$ are numbers. Therefore,

$$\begin{aligned}|p;n_1,n_2,\ldots,n_{N+3}) &= \frac{(B_1^+)^{n_1}(B_2^+)^{n_2}(b_3^+)^{n_3}(f_4^+)^{n_4}\ldots(f_{N+3}^+)^{n_{N+3}}}{\sqrt{n_1!n_2!n_3!}}|0\rangle\\ &= \sum_{k=0}^{n_1}\sum_{q=0}^{n_2} d(n_1,n_2,k,q)|p;n_1+n_2-k-q,k+q,n_3,...n_{N+3}\rangle,\end{aligned} \tag{7.20}$$

where again $d(n_1,n_2,k,q)$ are numbers. Clearly all vectors in the RHS have one and the same n_b and therefore they represent states with one and the same energy.

Next we simplify some of the notation:

$n_b \equiv b = n_1+n_2+n_3$ - the number of the bosons in a state $|p;n\rangle$,
$n_f \equiv f = n_4+...+n_{N+3}$ - the number of the fermions in $|p;n\rangle$.

We are now ready to describe the $so(3)$ structure of $V(N,p)$. The first step of the problem was already carried out (see (5.9)):

$$V(N,p) = \bigoplus_{n_b=\max(0,p-N)}^{p} V_1(N,p,n_b) \otimes V_2(N,p,n_f = p - n_b), \qquad (7.21a)$$

where $V_1(N,p,n_b)$ is an irreducible $gl(3)$ module, see (5.12), and $V_2(N,p,n_f)$, see (5.14) is an irreducible $gl(N)$ module. Another way to write (7.21a) is

$$V(N,p) = \bigoplus_{n_b=0}^{p} \Theta(N-p+n_b) V_1(N,p,n_b) \otimes V_2(N,p,n_f = p - n_b), \qquad (7.21b)$$

where $\Theta(x) = 0$ for $x < 0$ and 1 for $x \geq 0$.

Since $so(3)$ is a subalgebra of $gl(3)$, the rotation algebra transforms only the bosonic part $V_1(N,p,n_b)$ of $V_1(N,p,n_b) \otimes V_2(N,p,n_f = p - n_b)$. Hence in order to determine the angular momentum structure of the system in $V(N,p)$ one has to decompose each $gl(3)$ module $V_1(N,p,b)$ along the chain

$$gl(3) \supset so(3) \supset so(1). \qquad (7.22)$$

For the ladder representations of $gl(3)$, which we consider, the problem was solved in [58] directly for the quantum case. We shall use the results from [58], but without deformations.

One possible orthonormed basis in $V_1(N,p,b)$, consistent with (7.20) is

$$|p; n_1, n_2, n_3) = \frac{(B_1^+)^{n_1}(B_2^+)^{n_2}(B_3^+)^{n_3}}{\sqrt{n_1! n_2! n_3!}} |0\rangle, \quad b_k^-|0\rangle = 0, \quad n_1 + n_2 + n_3 = b = p - f. \qquad (7.23)$$

In this basis the $so(1)$ generator $\hat{S}_3$, see (7.16) p.37, is already diagonal. The basis (7.23) is not however reduced with respect to $so(3)$.

The decomposition of $V_1(N,p,b)$ into irreducible $so(3)$-modules reads [58]:

$$V_1(N,p,b) = \bigoplus_S V_1(N,p,b,S), \quad S = b, b-2, ..., 1 (\text{or } 0), \qquad (7.24)$$

where $V_1(N,p,b,S)$ is an irreducible $so(3)$ module with angular momentum S. As an appropriate orthonormed basis in $V_1(N,p,b,S)$ one can take

$$\begin{aligned} v(p,b,S,S_3) &= \sqrt{\frac{(b+S)!!(b-S)!!(S+S_3)!(S-S_3)!(2S+1)}{(b+S+1)!}} \\ &\times \sum_{x=\max(0,S_3)}^{\lfloor (S+S_3)/2 \rfloor} \sum_{y=0}^{(b-S)/2} (-1)^x \frac{\sqrt{(S_3+b-2x-2y)!(2x+2y)!!(2x+2y-2S_3)!!}}{(2x)!!(2y)!!(2x-2S_3)!!(S+S_3-2x)!(b-S-2y)!!} \\ &\times |p; x+y-S_3, x+y, b+S_3-2x-2y), \end{aligned} \qquad (7.25)$$

where S_3 is the projection of the angular momentum along the $z-$axes.

Observe that the coefficients in the RHS of (7.25) do not depend on p. Therefore if the relation

$$v(p,b,S,S_3) = \sum_{n_1,n_2,n_3} c(b,S,S_3,n_1,n_2,n_3)|p;n_1,n_2,n_3) \tag{7.26}$$

holds for a certain p, then it holds for any p.

As an orthonormed basis in $V(N,p)$ we take

$$v(p,b,S,S_3) \otimes (f_4)^{n_4}(f_5)^{n_5}...(f_{N+3})^{n_{N+3}}|0\rangle, \tag{7.27}$$

where $(f_4)^{n_4}(f_5)^{n_5}...(f_{N+3})^{n_{N+3}}|0\rangle$ with $n_4+...+n_{N+3}=n_f=p-n_b$ is an orthonormed basis in $V_2(N,p,n_f)$. Instead of (7.27)we shall also write

$$v(p,b,S,S_3) \otimes (f_4)^{n_4}(f_5)^{n_5}...(f_{N+3})^{n_{N+3}}|0\rangle = ||N,p,b,S,S_3,n_4,n_5,...,n_{N+3}\rangle\rangle. \tag{7.28}$$

The decomposition of any basis state (7.28) in terms of the S_3-basis (7.18) follows from (7.25):

$$\begin{aligned}
&||N,p,b,S,S_3,n_4,n_5,...,n_{N+3}\rangle\rangle = \sqrt{\frac{(b+S)!!(b-S)!!(S+S_3)!(S-S_3)!(2S+1)}{(b+S+1)!}} \\
&\times \sum_{x=\max(0,S_3)}^{\lfloor (S+S_3)/2 \rfloor} \sum_{y=0}^{(b-S)/2} (-1)^x \frac{\sqrt{(S_3+b-2x-2y)!(2x+2y)!!(2x+2y-2S_3)!!}}{(2x)!!(2y)!!(2x-2S_3)!!(S+S_3-2x)!(b-S-2y)!!} \\
&\times |p;x+y-S_3,x+y,b+S_3-2x-2y;n_4,n_5,...,n_{N+3}),
\end{aligned} \tag{7.29}$$

We call the basis (7.28) $SO(3)$*-reduced basis* or *an angular momentum basis*. By construction each basis vector $||N,p,b,S,S_3,n_4,n_5,...,n_{N+3}\rangle\rangle$ is an eigenvector of $\hat{H}$, $\hat{\mathbf{S}}^2$ and $\hat{S}_3$:

$$\begin{aligned}
&\hat{H}||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle = \frac{\hbar\omega}{|N-3|}(Nn_b+3n_f)||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle, \\
&\hat{\mathbf{S}}^2||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle = S(S+1)||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle, \\
&\hat{S}_3||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle = S_3||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle.
\end{aligned}$$

The admissible values of $n_4,...,n_{N+3}$ distinguish between the basis states with the same energy, angular momentum and its third projection. Let us summarize.

Corollary 7.1: *A reduced basis vector* $||N, p, b, S, S_3, n_4, n_5, ..., n_{N+3}\rangle\rangle$ *corresponds to a state of the system with energy* $E = \omega\hbar(3p + Nb - 3b)/|N-3|$*, angular momentum* S*, its third projection* S_3 *and fermionic numbers* $n_4, ..., n_{N+3}$*. All different states* $||N, p, b, S, S_3, n_4, n_5, ..., n_{N+3}\rangle\rangle$*, namely those with*

(a) $b = \max(0, p-N), \max(0, p-N)+1, ..., p-1, p,$

(b) $S = b, b-2, ..., 1(or\ 0),$

(c) $S_3 = -S, -S+1, ..., S,$

(d) fermionic numbers $n_4, n_5, ..., n_{N+3}$ *such that* $n_4 + ... + n_{N+3} = p - b,$

constitute an orthonormed basis in the state space $V(N, p)$*.*

Let us give an example.

Example 7.1. Let $N = 2$ and $p = 1$. The state space is 5 dimensional. The angular momentum basis read:

$$||N=2, p=1, b=1, S=1, S_3=1,0,-1, n_4=0, n_5=0\rangle\rangle, \tag{7.30a}$$

$$||N=2, p=1, b=0, S=0, S_3=0, n_4=1, n_5=0\rangle\rangle. \tag{7.30b}$$

$$||N=2, p=1, b=0, S=0, S_3=0, n_4=0, n_5=1\rangle\rangle. \tag{7.30c}$$

In terms of the S_3-basis (7.17) and the initial basis the above states read (we skip the common for all states labels $N = 2$ and $p = 1$):

$$\begin{aligned} &||b=1, S=1, S_3=1, n_4=0, n_5=0\rangle\rangle = -|0,1,0,0,0) \\ &\quad = -\frac{1}{\sqrt{2}}|1,0,0,0,0\rangle - \frac{\mathrm{i}}{\sqrt{2}}|0,1,0,0,0\rangle, \end{aligned} \tag{7.31a}$$

$$||b=1, S=1, S_3=0, n_4=0, n_5=0\rangle\rangle = |0,0,1,0,0) = |0,0,1,0,0\rangle, \tag{7.31b}$$

$$\begin{aligned} &||b=1, S=1, S_3=-1, n_4=0\rangle\rangle = |p=1; 1,0,0,0,0) \\ &\quad = \frac{1}{\sqrt{2}}\Big(|1,0,0,0,0\rangle - \mathrm{i}|0,1,0,0,0\rangle\Big), \end{aligned} \tag{7.31c}$$

$$||b=0, S=0, S_3=0, n_4=0, n_5=1\rangle\rangle = |0,0,0,0,1) = |0,0,0,0,1\rangle, \tag{7.31d}$$

$$||b=0, S=0, S_3=0, n_4=1, n_5=0\rangle\rangle = |0,0,0,1,0) = |0,0,0,1,0\rangle. \tag{7.31e}$$

There are 5 states. The first three of them correspond to orbital momentum 1. The last two states have orbital momentum 0. For an example corresponding to any N and $p = 1, 2, 3$ see appendix D.

We recall, see (3.29), that the angular momentum of the particles is measured in units $\frac{\hbar}{N-3}$, whereas for the entire system it is $\hat{M}_j = \frac{\hbar N}{N-3}\hat{S}_j$.

The last physical observable which we are going to consider is the parity operator P, called also space inversion operator. In the $3D$ space this operator transforms the frame vectors $\mathbf{e}_k$ into their mirror images:

$$\mathrm{P}\mathbf{e}_k = -\mathbf{e}_k, \quad k = 1, 2, 3. \tag{7.32}$$

In a matrix form

$$\mathrm{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7.33}$$

The matrix P is an orthogonal matrix and therefore $\mathrm{P} \in O(3)$. In fact [57]

$$O(3) = \{g, -g = gP | g \in SO(3)\}. \tag{7.34}$$

Our problem is to find out how P acts in the state space $V(N, p)$. We shall use the circumstance that P is an element also from the group $GL(3)$, $\mathrm{P} \in GL(3)$. This is evident since $GL(3)$ is the collection of all 3×3 matrices with determinant different from zero. As a second step we use the exponential map: [59] if $x \in gl(3)$, then $\exp x \in GL(3)$ in order to find (first in the defining 3×3 representation) which is the element x from the algebra $gl(n)$ for which $\exp x = \mathrm{P} \in GL(3)$.

To begin with take the element $-\mathrm{i}\varphi(E_{11}+E_{22}+E_{33})$ from $gl(3)$. Then (10302) $g(\varphi) = \exp(-\mathrm{i}\varphi(E_{11} + E_{22} + E_{33})) \in GL(3)$. Therefore

$$g(\varphi) = \exp(-\mathrm{i}\varphi(E_{11} + E_{22} + E_{33})) = \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!}(E_{11} + E_{22} + E_{33})^k. \tag{7.35}$$

But in the defining (3×3) representation $E_{11} + E_{22} + E_{33} = E$ is the 3×3 unit matrix E and $E^k = E$. Consequently

$$g(\varphi) = \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!} E = E \exp(-\mathrm{i}\varphi), \tag{7.36}$$

and for $\varphi = \pi$ (7.36) yields: $g(\pi) = E \exp(-\mathrm{i}\pi) = E \cos \pi = -E = P$. Thus,

$$\mathrm{P} = \exp(-\mathrm{i}\pi(E_{11} + E_{22} + E_{33})). \tag{7.37}$$

The relevance of the last relation stems from the observation that it holds in any representation of the Lie algebra $gl(3)$ and we know the realization of the $gl(3)$ generators E_{kk} in any state space $V(N, p)$. Indeed according to (4.7) $E_{ij} = b_i^+ b_j^-$, $i, j = 1, 2, 3$, and therefore

$$\mathrm{P} = \exp(-i\pi(b_1^+ b_1^- + b_2^+ b_2^- + b_3^+ b_3)). \tag{7.38}$$

Taking into account that $b_k^+ b_k^-$ are number operators, see (4.5c), we find:

$$\mathrm{P}|p,n\rangle = \exp(-i\pi(n_1+n_2+n_3))|p,n\rangle = (-1)^{n_1+n_2+n_3}|p,n\rangle. \tag{7.39}$$

We see that the basis vectors $|p;n\rangle$ are eigenvectors of the parity operator and since $n_1 + n_2 + n_3 = b$ we conclude:

$$\mathrm{P}|p,n\rangle = (-1)^{n_1+n_2+n_3}|p,n\rangle = (-1)^b|p,n\rangle. \tag{7.40}$$

A number of consequences follow from (7.40).

Corollary 7.2. *Any state $|p;n\rangle$ is invariant under the action of the parity operator.*

Obviously $\mathrm{P}^2 = 1$ and therefore $\mathrm{P} = \mathrm{P}^{-1}$. Moreover for any two basis states $|p;n\rangle$ and $|p;n'\rangle$

$$(\mathrm{P}|p;n\rangle, |p;n'\rangle) = (|p;n\rangle, \mathrm{P}|p;n'\rangle), \quad (\mathrm{P}|p;n\rangle, \mathrm{P}|p;n'\rangle) = (|p;n\rangle, |p;n'\rangle). \tag{7.41}$$

Corollary 7.3. P *is an unitary Hermitian operator.*

The observation that the energy of a state $|p;n\rangle$ from $V(N,p)$ is in one to one correspondence with b, see (5.4), yields:

Corollary 7.4. *All states $|p;n)$ from $V(N,p)$ which have one and the same energy have also one and the same parity $(-1)^b$. In particular all states from $V(N,p,b,f)$ have parity $(-1)^b$. Consequently the parity of the state $||N,p,b,S,S_3,n_4,n_5,...,n_{N+3}\rangle\rangle$, see (7.28), is also $(-1)^b$.*

Corollary 7.5. *It is straightforward to verify that*

$$\mathrm{P}\hat{H}\mathrm{P}^{-1} = \hat{H}, \;\; \Leftrightarrow \;\; [\mathrm{P}, H] = 0, \tag{7.42a}$$

$$\mathrm{P}\hat{r}_{\alpha k}\mathrm{P}^{-1} = -\hat{r}_{\alpha k}, \;\; k = 1,2,3, \tag{7.42b}$$

$$\mathrm{P}\hat{p}_{\alpha k}\mathrm{P}^{-1} = -\hat{p}_{\alpha k}, \;\; k = 1,2,3, \tag{7.42c}$$

$$\mathrm{P}\hat{S}_k\mathrm{P}^{-1} = \hat{S}_k, \;\; k = 1,2,3, \tag{7.42d}$$

Thus the parity operator P *is an integral of motion, the Hamiltonian $\hat{H}$ is a proper scalar operator, the position and the momentum operators are proper vector operators, whereas the angular momentum is pseudovector operator.*

So far we have clarified what are the possible nests for any particle whenever the system is in a basis state $|p; n\rangle$. What we have not clarified yet is what is the probability the particle to occupy one of the nests. This issue will be the topic of the next discussion. We will show that with one and the same probability the particle under consideration can be in anyone of its nests.

We begin with the states from the Class II. If the system is in a state $|p; n\rangle$ from this class, then the measurements of the coordinates of the $\alpha-$th particle yield that its nests are on a sphere which form the vertices of a rectangular parallelepiped

$$r_{\alpha 1} = \pm\sqrt{n_1 + n_{\alpha+3}}, \quad r_{\alpha 2} = \pm\sqrt{n_2 + n_{\alpha+3}}, \quad r_{\alpha 3} = \pm\sqrt{n_3 + n_{\alpha+3}} k = 1, 2, 3. \qquad (7.43)$$

First we derive a few preliminary results.

Proposition 7.2. *Any state $|p; n\rangle$ from Class II is invariant under rotation on angle π around $x-, y-$ or $z-$ axes.*

Proof. Consider a rotation of the system on an angle π around $z-$axes. Since any state $|p; n\rangle$ is defined up to a multiplicative constant, we have to show that $|p; n\rangle$ is an eigenstate of the rotation operator $\exp(\mathrm{i}\pi \hat{S}_3)$. To this end we expand the state $|p; n\rangle$ in the basis $\{|p; n)\}$, namely via the eigenvectors of $\hat{S}_3$. Using relations (7.12) we calculate:

$$|p; n_1, n_2, ...\rangle = \sum_{k=0}^{n_1} \sum_{q=0}^{n_2} c(n_1, n_2, k, q) |p; n_1 + n_2 - k - q, k + q, n_3, ..., n_{N+3}), \qquad (7.44)$$

where $c(n_1, n_2, k, q)$ are numbers. Then from (7.16) and taking into account that $\exp(\mathrm{i}\pi) = -1)$, we calculate:

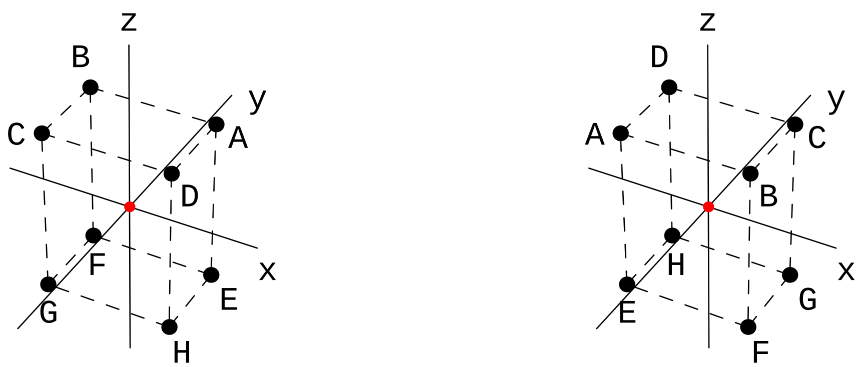
$$\begin{aligned} &\exp(\mathrm{i}\pi \hat{S}_3)|p; n_1, n_2, ...\rangle \\ &\sum_{k=0}^{n_1} \sum_{q=0}^{n_2} c(n_1, n_2, k, q) \exp(\mathrm{i}\pi(2k + 2q - n_1 - n_2))|p; n_1 + n_2 - k - q, k + q, n_3, ..., n_{N+3}) \,. \\ &= (-1)^{n_1+n_2}|p; n_1, n_2, ...\rangle \end{aligned} \qquad (7.45)$$

In a similar way, a replacement in (7.12) $1 \to 2$, $2 \to 3$, $3 \to 1$, (resp $1 \to 3$, $2 \to 1$, $3 \to 2$) diagonalizes $\hat{S}_1$ (resp. $\hat{S}_2$). Then repeating the above arguments one proves that any state $|p; n\rangle$ is invariant under rotation around $x-$ or around $y-$axes on angle π. This completes the proof.

Proposition 7.3. *Let the system be in a state* $|p; n\rangle$ *from the Class II. Then the* $\alpha-th$ *particle will occupy with equal probability any one of the corresponding nests.*

Proof. Assume for definiteness that $n_1 \neq 0$ and $n_2 \neq 0$. There are two cases to be considered.

Figures 4



a) $n_3 + n_{\alpha+3} \neq 0$. The configuration of the nests is shown on the LHS of Figure 4.
We have denoted by A the nest with all positive coordinates:

$$A = (\sqrt{n_1 + n_{\alpha+3}}, \sqrt{n_2 + n_{\alpha+3}}, \sqrt{n_3 + n_{\alpha+3}}). \tag{7.46}$$

The coordinates of the rest 7 nests are also clear. They differ only by signs from those of A.

Perform now a rotation on angle π around $z-$axes. Clearly it will bring the system in a configuration shown on the RHS of Figure 4. But according to proposition 7.2 both space configurations shown on Figure 4 represent one and the same state. This in particular means that

$$P(A) = P(C),\ P(B) = P(D),\ P(E) = P(G),\ P(F) = P(H), \tag{7.47}$$

where $P(X)$ is the probability the (α-th) particle to be in the nest X, $X = A, B, ...H$.

In a similar way, performing a rotation on angle π around $x-$axes and using again proposition 7.2, one concludes that

$$P(A) = P(H),\ P(D) = P(E),\ P(B) = P(G),\ P(C) = P(F) \tag{7.48}$$

which together with (7.47) yields:

$$P(A) = P(C) = P(F) = P(H), \quad P(B) = P(D) = P(E) = P(G). \tag{7.49}$$

A rotation around $y-$axes on angle π does not give anything new. As a next step we use the circumstance that the state $|p; n\rangle$ is invariant also with respect to parity transformation, see (7.40), which yields:

$$P(A) = P(G),\ P(B) = P(H),\ P(C) = P(E),\ P(D) = P(F). \tag{7.50}$$

The latter together with (7.48) proves proposition 7.3:

$$P(A) = P(B) = P(C) = P(D) = P(E) = P(F) = P(G) = P(H). \tag{7.51}$$

b) $n_3 + n_{\alpha+3} = 0$. In this case the nests are only 4 and all of them are in the xOy plane. The proof of the equal probability given above works perfectly well also in this case and is even simpler.

Let us turn to Class III configurations.

Proposition 7.4. *Let the system be in a state $|p; n\rangle$ from the Class III. Then the $\alpha-$th particle will occupy with equal probability any one of the allowed nests.*

Proof. Consider for definiteness the case with $n_1 = n_2 = 0, n_3 \neq 0$. If $n_{\alpha+3} = 0$ the space configuration has only two nests, $\pm\sqrt{n_3}\mathbf{e}_3$, see (6.88a). The structure is the same as in Figure 3: there are two nests. The equal probability the particle to be in one of them is a consequence of proposition 6.4

The other opportunity is $n_{\alpha+3} = 1$, i.e. the state is $\Psi = |p; 0, 0, n_3, ..., 1_{\alpha+3}, ..\rangle$. The space configuration is similar to the one on Figure 5. In cylindrical coordinates (ρ, α, z) the nests consist of all points with

$$\rho = \sqrt{2}, \quad 0 \leq \alpha < 2\pi, \quad z = \pm\sqrt{n_3 + 1}. \tag{7.52}$$

It is straightforward to verify that $\hat{S}_3\Psi = 0$ and therefore

$$e^{i\varphi\hat{S}_3}\Psi = \Psi, \tag{7.53}$$

i.e, the state Ψ is invariant under any rotations around $z-$axes. Let $P(\sqrt{2}, \varphi, \pm\sqrt{n_3+1})$ be the probability density the particle to be in the nest $(\sqrt{2}, \varphi, \pm\sqrt{n_3+1})$. Then from the rotation invariance around z one concludes that the probability density does not depend on φ, whereas the parity invariance yields in addition that $P(\sqrt{2}, \varphi, \pm\sqrt{n_3+1})$ is independent on the sign in front of z. Then from the normalization condition for the probability density one finds $P = 1/\sqrt{32}\pi$.

Finally we consider the Class I configurations. Then any point from a sphere with a radius $\sqrt{3n_{\alpha+3}}$ is a nest for the $\alpha$the particle.

Proposition 7.5. *Let the system be in a state $\Psi = |p; 0, 0, 0; n_4, ..., n_{\alpha+3}, ..\rangle$ from Class I. Then the αth particle will occupy with equal probability any one of the allowed nests.*

Proof. If $n_{\alpha+3} = 0$ the αth particle is "sitting" on the center of mass with probability 1. So assume that $n_{\alpha+3} = 1$. It is straightforward to verify that $\hat{S}_k\Psi = 0$ for any $k = 1, 2, 3$. Therefore

$$e^{i\varphi\hat{S}_k}\Psi = \Psi, \quad k = 1, 2, 3, \tag{7.54}$$

i.e, the state Ψ is invariant under rotations around $x-$, $y-$ or $z-$axes. Hence it is invariant under an arbitrary rotation. Since any point a from this sphere can be moved by an appropriate rotation onto any other point b also from this sphere, the probability density is one and the same for any point on the sphere with radius $\sqrt{3}$. Then the normalization condition yields $P(a) = 1/12\pi$ for any nest a of the sphere.

Appendix A: Proof of Proposition 6.5.

The transformation relations of the operators $(\hat{r}_{\alpha 1}, \hat{r}_{\alpha 2}, \hat{r}_{\alpha 3})$ under global rotations g are the same as for the frame vectors:

$$\hat{r}(g)_{\alpha i} = \hat{U}(g)\hat{r}_{\alpha i}\hat{U}(g)^{-1} = \sum_{j=1}^{3} \hat{r}_{\alpha j} g_{ji}, \quad i = 1, 2, 3, \tag{A.1}$$

Proof. Since the results to be proved are the same for any of the particles, in what follows we skip the subscript α. We consider first a rotation $g(\mathbf{e}_3, \varphi)$, namely a rotation around $\mathbf{e}_3$ on angle φ. Then

$$\hat{r}(g(\mathbf{e}_3, \varphi))_j = \hat{V}(g(\mathbf{e}_3, \varphi))\hat{r}_j = e^{-i\varphi\hat{S}_3}\hat{r}_j e^{i\varphi\hat{S}_3}, \quad j = 1, 2, 3. \tag{A.2}$$

Since $[S_3, \hat{r}_3] = 0$, the above relation yields

$$\hat{r}(g(\mathbf{e}_3, \varphi))_3 = \hat{r}_3. \tag{A.3}$$

The computation of $\hat{r}(g(\mathbf{e}_3, \varphi))_j$ for $j = 1, 2$, is not that simple. Introduce first the eigenvectors of S_3 (the weight vectors of the Cartan subalgebra), namely

$$\hat{r}_+ = \hat{r}_1 + i\hat{r}_2, \quad \text{and} \quad \hat{r}_- = \hat{r}_1 - i\hat{r}_2. \tag{A.4}$$

Then

$$[S_3, \hat{r}_+] = \hat{r}_+, \quad \text{and} \quad [S_3, \hat{r}_-] = -\hat{r}_-. \tag{A.5}$$

Next we compute $e^{-i\varphi\hat{S}_3}\hat{r}_\pm e^{i\varphi\hat{S}_3}$ using the Backer-Campbell-Hausdorf formula (as given in [56], which in this case reads:

$$e^{-i\varphi\hat{S}_3}\hat{r}_\pm e^{i\varphi\hat{S}_3} = \sum_{k=0}^{\infty} \frac{1}{k!}[-i\varphi S_3, \hat{r}_\pm]_{(k)}, \tag{A.6}$$

where the multiple commutator is defined as

$$\begin{aligned} &[A, B]_{(k)} = [A, [A, B]_{(k-1)}], \\ &[A, B]_{(1)} = [A, B] = AB - BA, \\ &[A, B]_{(0)} = B. \end{aligned} \tag{A.7}$$

Then

$$[-i\varphi\hat{S}_3, \hat{r}_\pm]_{(k)} = (\mp i\varphi)^k \hat{r}_\pm. \tag{A.8}$$

Inserting (A.8) in (A.6) one obtains:

$$\begin{aligned} e^{-i\varphi\hat{S}_3}\hat{r}_+ e^{i\varphi\hat{S}_3} &= \sum_{k=0}^{\infty} \frac{(-i\varphi)^k}{k!}\hat{r}_+ = e^{-i\varphi}\hat{r}_+, \\ e^{-i\varphi\hat{S}_3}\hat{r}_- e^{i\varphi\hat{S}_3} &= \sum_{k=0}^{\infty} \frac{(i\varphi)^k}{k!}\hat{r}_- = e^{i\varphi}\hat{r}_- \end{aligned} \tag{A.9}$$

And finally a replacement in (A.9) of $\hat{r}_\pm$ with $\hat{r}_1,\ \hat{r}_2$ according to (A.4) yields:

$$\begin{aligned} \hat{r}(g(\mathbf{e}_3,\varphi))_1 &= e^{-i\varphi\hat{S}_3}\hat{r}_1 e^{i\varphi\hat{S}_3} = \hat{r}_1\cos\varphi + \hat{r}_2\sin\varphi, \\ \hat{r}(g(\mathbf{e}_3,\varphi))_2 &= e^{-i\varphi\hat{S}_3}\hat{r}_2 e^{i\varphi\hat{S}_3} = -\hat{r}_1\sin\varphi + \hat{r}_2\cos\varphi, \\ \hat{r}(g(\mathbf{e}_3,\varphi))_3 &= e^{-i\varphi\hat{S}_3}\hat{r}_3 e^{i\varphi\hat{S}_3} = \hat{r}_3 \end{aligned} \tag{A.10}$$

The last result can be written in a compact form,

$$\hat{r}(g(\mathbf{e}_3,\varphi))_j = \hat{V}(g(\mathbf{e}_3,\varphi))\hat{r}_j = \sum_{k=1}^{3}\hat{r}_k g(\mathbf{e}_3,\varphi)_{k,j} = (\hat{r}g(\mathbf{e}_3,\varphi))_j, \tag{A.11}$$

In the derivation of (A.11) we have used only the commutation relations $[\hat{S}_3,\hat{r}_j]$ which are invariant under cyclic change

$$1\rightarrow 3,\qquad 2\rightarrow 1,\quad 3\rightarrow 2. \tag{A.12}$$

Therefore Eqs. (A.10) remain true under the change (A.12):

$$\begin{aligned} \hat{r}(g(\mathbf{e}_2,\varphi))_3 &= e^{-i\varphi\hat{S}_2}\hat{r}_3 e^{i\varphi\hat{S}_2} = \hat{r}_3\cos\varphi + \hat{r}_1\sin\varphi, \\ \hat{r}(g(\mathbf{e}_2,\varphi))_1 &= e^{-i\varphi\hat{S}_2}\hat{r}_1 e^{i\varphi\hat{S}_2} = -\hat{r}_3\sin\varphi + \hat{r}_1\cos\varphi, \\ \hat{r}(g(\mathbf{e}_2,\varphi))_2 &= e^{-i\varphi\hat{S}_2}\hat{r}_2 e^{i\varphi\hat{S}_2} = \hat{r}_2, \end{aligned}$$

which yields

$$\hat{r}(g(\mathbf{e}_2,\varphi))_j = \hat{V}(g(\mathbf{e}_2,\varphi))\hat{r}_j = \sum_{k=1}^{3}\hat{r}_k g(\mathbf{e}_2,\varphi)_{k,j} = (\hat{r}g(\mathbf{e}_2,\varphi))_j. \tag{A.13}$$

Similarly, from (A.12) and (A.13) we derive

$$\hat{r}(g(\mathbf{e}_1,\varphi))_j = \hat{V}(g(\mathbf{e}_1,\varphi))\hat{r}_j = \sum_{k=1}^{3}\hat{r}_k g(\mathbf{e}_1,\varphi)_{k,j} = (\hat{r}g(\mathbf{e}_1,\varphi))_j, \tag{A.14}$$

Equations (A.11), (A.13), (A.14) can be unified:

$$\hat{r}(g(\mathbf{e}_i,\varphi_i))_j = \hat{V}(g(\mathbf{e}_i,\varphi_i))\hat{r}_j = \sum_{k=1}^{3}\hat{r}_k g(\mathbf{e}_i,\varphi_i)_{k,j} = (\hat{r}g(\mathbf{e}_i,\varphi))_j. \tag{A.15}$$

We recall that $\hat{V}(g)$ gives a representation of $SO(3)$. Therefore

$$\begin{aligned} &\hat{r}(g)_i = V(g)\hat{r}_i = V(g(\mathbf{e}_3,\alpha)g(\mathbf{e}_2,\beta)g(\mathbf{e}_3,\gamma))\hat{r}_{\mathrm{i}} \\ &= V(g(\mathbf{e}_3,\alpha))V(g(\mathbf{e}_2,\beta))V(g(\mathbf{e}_3,\gamma))\hat{r}_i \\ &= \sum_{l,k,j}\hat{r}_l g(\hat{\mathbf{r}}_3,\alpha)_{lk} g(\hat{\mathbf{r}}_2,\beta)_{kj} g(\hat{\mathbf{r}}_3,\gamma)_{ji} \\ &= \sum_{l}\hat{r}_l g(\mathbf{e}_3,\alpha)g(\mathbf{e}_2,\beta)g(\mathbf{e}_3,\gamma))_{li} = \sum_{l}\hat{r}_l g(\alpha,\beta.\gamma)_{li} = (\hat{r}g)_i. \end{aligned} \tag{A.16}$$

The above result holds for any particle. In particular for the αth particle one has:

$$\hat{r}(g)_{\alpha i} = \sum_{j=1}^{3} \hat{r}_{\alpha j} g_{ji} = (\hat{r}_\alpha g)_i. \tag{A.17}$$

This completes the proof.

Appendix B: Proof of Proposition 6.6

If the system is in a basis state $|p;n\rangle$ from Class II, then the nests (6.83) are the only nests for the αth particle.

Proof. We know from corollary 6.2 that if the dispersion $\mathrm{Disp}(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle}$ vanishes for a certain g and for all $k = 1, 2, 3$ then the set

$$\Gamma(|p;n\rangle), \alpha, g) = \{r(g)_{\alpha,1}\mathbf{e}(g)_1 + r(g)_{\alpha,2}\mathbf{e}(g)_2 + r(g)_{\alpha,3}\mathbf{e}(g)_3\}. \tag{B.1}$$

determines admissible places, i.e., nests for the $\alpha$the particle. We proceed to prove that the nests (B.1) coincide with those in (6.83)

The first task to solve is to determine all 3×3 orthogonal matrices g for which the dispersion $D(\hat{r}(g)^2_{\alpha,k})_{|p;n\rangle}$ vanishes. According to (6.60) we have to find all solutions of the equation

$$\begin{aligned} &g^2_{1k} g^2_{2k}(2n_1n_2 + n_1 + n_2) + g^2_{1k} g^2_{3k}(2n_1n_3 + n_1 + n_3) \\ &+ g^2_{3k} g^2_{2k}(2n_2n_3 + n_2 + n_3) = 0, \quad k = 1, 2, 3, \quad \alpha = 1, ..., N. \end{aligned} \tag{B.2}$$

where the unknown are the matrix elements of g. Since for all states from Class II

$$2n_2n_3 + n_2 + n_3 \neq 0, \quad 2n_1n_3 + n_1 + n_3 \neq 0, \quad 2n_1n_2 + n_1 + n_2 \neq 0, \tag{B.3}$$

the problem reduces to determine all solutions of the equations

$$g^2_{1k} g^2_{2k} = 0, \quad g^2_{1k} g^2_{3k} = 0, \quad g^2_{2k} g^2_{3k} = 0, \quad k = 1, 2, 3. \tag{B.4}$$

Here are the main steps in solving the problem.

Find first all matrices which satisfy the restriction $g_{23}g_{33} = 0$. It yields two classes of matrices:

Class 1. All matrices $g(\alpha, \beta, \gamma)$ with α, γ being arbitrary and $\beta = 0, \pi/2, \pi$,

Class 2. All matrices $g(\alpha, \beta, \gamma)$ with β, γ being arbitrary and $\alpha = 0, \pi$.

The conditions $g_{21}g_{31} = 0$ and $g_{22}g_{32} = 0$ do not lead to additional restrictions on Class 1. So we have:

1A. All matrices $g(\alpha, \beta = 0, \gamma)$, which is a rotation of angle $\alpha + \gamma$ about z axes.

1B. All matrices $g(\alpha, \beta = \pi, \gamma)$, which is a rotation of angle $\alpha - \gamma$ about z axes.

1c. All matrices $g(\alpha, \beta = \pi/2, \gamma)$.

The equations $g_{1k}^2 g_{3k}^2 = 0, \quad k = 1, 2, 3$ put additional restrictions only on the class 1c:

1C. All matrices $g(\alpha, \beta = \pi/2, \gamma)$ with $\gamma = 0, \pi/2, \pi, 3p/2$.

For further use we collect part of the results obtained so far.

Corollary B.1. *All solutions of the equations $g_{2k}g_{3k} = 0$ and $g_{1k}g_{3k} = 0, \quad k = 1, 2, 3,$ are given with the $g(\alpha, \beta, \gamma)$ matrices from the subclasses 1A, 1B, 1C, defined above (see below: the Class 2 does not contain new solutions).*

Finally the equations $g_{1k}^2 g_{2k}^2 = 0, \quad k = 1, 2, 3$ lead to the following solutions:

1Aa

$$g(\alpha, \beta = 0, \gamma) = \begin{pmatrix} \cos(\alpha + \gamma), & -\sin(\alpha + \gamma), & 0 \\ \sin(\alpha + \gamma), & \cos(\alpha + \gamma), & 0 \\ 0, & 0 & 1 \end{pmatrix}, \ \alpha + \gamma = 0, \pi/2, \pi, 3\pi/2, \qquad (B.5a)$$

1Ba.

$$g(\alpha, \beta = \pi, \gamma) = \begin{pmatrix} -\cos(\alpha - \gamma), & -\sin(\alpha - \gamma), & 0 \\ -\sin(\alpha - \gamma), & \cos(\alpha - \gamma), & 0 \\ 0, & 0 & -1 \end{pmatrix}, \ \alpha - \gamma = 0, \pi/2, \pi, 3p/2. \quad (B.5b)$$

1Ca.

$$g(\alpha, \beta = \pi/2, \gamma) = \begin{pmatrix} -\sin\alpha \sin\gamma, & -\sin\alpha\cos\gamma, & \cos\alpha \\ \cos\alpha\sin\gamma, & \cos\alpha\cos\gamma, & \sin\alpha \\ -\cos\gamma, & \sin\gamma & 0 \end{pmatrix}, \quad \alpha, \gamma = 0, \pi/2, \pi, 3p/2. \qquad (B.5c)$$

The conditions $g_{2k}g_{3k} = 0$ is satisfied by the following matrices from Class 2:

Class 2A: all $g(\alpha = 0, \pi, \beta = 0, \pi/2, \pi, \gamma = 0, \pi/2, \pi, 3\pi/2)$,

Class 2B: all $g(\alpha = 0, \pi, \beta = 0, \pi, \gamma)$.

Clearly the solutions from the Class 2A and Class 2B are particular cases of solutions from the classes 1A, 1B and 1C and therefore we do not consider them anymore.

Thus if the system is in a state $|p; n\rangle$, for which conditions (B.3) holds, then the classes 1Aa, 1Ba and 1Ca determine all g matrices for which the dispersion of $\hat{r}(g)^2_{\alpha k}$ along $e(g)_k$, $k = 1, 2, 3$ vanishes. Therefore for any such g the set

$$\Gamma(|p; n\rangle, \alpha, g) = \sum_{k=1}^{3} r(g)_{\alpha,k} \mathbf{e}(g)_k \tag{B.6}$$

determines a set of nests for the αth particle, where $r(g)^2_{\alpha k}$ is an eigenvalue of $\hat{r}(g)^2_{\alpha k}$ on $|p; k\rangle$:

$$r(g)^2_{\alpha,k} = g^2_{1k}(n_1 + n_{\alpha+3}) + g^2_{2k}(n_2 + n_{\alpha+3}) + g^2_{3k}(n_3 + n_{\alpha+3}), \quad k = 1, 2, 3. \tag{B.7}$$

and $r(g)_{\alpha,k} = \pm\sqrt{r(g)^2_{\alpha,k}}$.

At this place we shall make use of the following property of the matrices (B.2):

Corollary B.2 *Let g be any matrix from the classes 1Aa,1Ba or 1Ca. Then each row and each column of g consist of two zeros and one number ± 1.*

Then in view of the above property only one term in the RHS of (B.7) survives,

$$r(g)^2_{\alpha,k} = (n_{j_k} + n_{\alpha+3}) \quad \Longrightarrow \quad r(g)_{\alpha,k} = \pm\sqrt{n_{j_k} + n_{\alpha+3}} = r_{\alpha,j_k}, \tag{B.8}$$

where $\{j_1, j_2, j_3\}$ is a permutation of $\{1, 2, 3\}$ Similarly

$$\mathbf{e}(g)_k = \mathbf{e}_{j_k} g_{j_k,k}, \tag{B.9}$$

where $g_{j_k,k} = 1$ or -1. Therefore

$$\Gamma(|p; n\rangle), \alpha, g) = \{\sum_k r(g)_{\alpha,k} \mathbf{e}(g)_k\} = \sum_{j=1}^{3} \pm\sqrt{n_j + n_{\alpha+3}}\ \mathbf{e}_j = \Gamma(|p; n\rangle, \alpha), \tag{B.10}$$

see (6.24). This completes the proof.

Appendix C: Proof of Proposition 6.8.

Below we prove the third part of proposition 6.8, namely we show that the nests of the αth particle, whenever the system is in the state $|p; 0, 0, n_3, .., 1_{\alpha+3}, ..\rangle$ are (6.91). The rest namely the equations (6.89) and (6.90)are proved in a similar way.

Consider first the solution 1A , see corollary B.1. Without loss of generality we set $\beta = \gamma = 0$, leaving α to be arbitrary, i.e., we have

$$g(\alpha, \beta = 0, \gamma = 0) = \begin{pmatrix} \cos\alpha, & -\sin\alpha, & 0 \\ \sin\alpha, & \cos\alpha, & 0 \\ 0, & 0 & 1 \end{pmatrix}, \tag{C.1}$$

which is a rotation about the $z-$axes on angle α. Then (6.87)yields the following eigenvalues $r(g)^2_{\alpha k}$ of $\hat{r}(g)^2_{\alpha k}$ on $|p; 0, 0, n_3, ..., n_{\alpha+3}, ..\rangle$:

$$r(g)^2_{\alpha 1} = 1, \quad r(g)^2_{\alpha 2} = 1, \quad r(g)^2_{\alpha 3} = n_3 + 1. \tag{C.2}$$

Equation (6.68)tells us that the nests of the αth particle corresponding to $g(\alpha, \beta = 0, \gamma = 0)$, see (C.1), whenever the system is in the state $|p; 0, 0, n_3, ..., 1_{\alpha+3}, ..\rangle$ are:

$$\Gamma\big(|p; 0, 0, n_3, ..., 1_{\alpha+3}, ..\rangle,\ g(\alpha, \beta = 0, \gamma = 0),\ \alpha\big) = \xi_1 \mathbf{e}(g)_1 + \xi_2 \mathbf{e}(g)_2 + \xi_3 \sqrt{n_3 + 1}\mathbf{e}(g)_3, \tag{C.3}$$

where $\xi_1,\ \xi_2,\ \xi_3 = \pm 1$.

In the initial basis $\mathbf{e} = g\mathbf{e}(g)$ the nests (C.3) read

$$\begin{aligned} &\Gamma\big(|p; 0, 0, n_3, ..., 1_{\alpha+3}, ..\rangle,\ g(\alpha, \beta = 0, \gamma = 0),\ \alpha\big) = \big\{(\xi_1 \cos\alpha - \xi_2 \sin\alpha)\mathbf{e}_1 \\ &+ (\xi_1 \sin\alpha + \xi_2 \cos\alpha)\mathbf{e}_2 + \xi_3 \sqrt{n_3 + 1}\mathbf{e}_3 \mid \alpha \in \mathbf{R}\big\}. \end{aligned} \tag{C.4}$$

The first impression might be that the nests corresponding to different choices of $\xi_1,\ \xi_2,\ \xi_3 = \pm 1$ are different. This is however not the case. With elementary considerations one shows that the different choices of $\xi_1,\ \xi_2$ lead to one and the same collection of nests. If for instance $\xi_1 = 1,\ \xi_2 = 1$, then the RHS of (C.4) reads:

$$(\cos\alpha - \sin\alpha)\mathbf{e}_1 + (\sin\alpha + \cos\alpha)\mathbf{e}_2 + \xi_3 \sqrt{n_3 + 1}\mathbf{e}(g)_3. \tag{C.5}$$

Replacing in (C.5)α with $\alpha + \pi/2$ one relabels the nests but does not change the collection of them. After this substitution (C.5)reads

$$(-\cos\alpha - \sin\alpha)\mathbf{e}_1 + (-\sin\alpha + \cos\alpha)\mathbf{e}_2 + \xi_3 \sqrt{n_3 + 1}\mathbf{e}(g)_3. \tag{C.6}$$

which corresponds to the choice $\xi_1 = -1,\ \xi_2 = 1$ in (C.4). Hence the choices $\xi_1 = 1,\ \xi_2 = 1$ and $\xi_1 = -1,\ \xi_2 = 1$ describe one and the same collection of nests. In a similar way one concludes that also the other two choices $\xi_1 = 1,\ \xi_2 = -1$ and $\xi_1 = -1,\ \xi_2 = -1$ give the

same nests as the choice $\xi_1 = 1,\ \xi_2 = 1$. Thus the nests of the αth particle corresponding to solution 1A, namely to the matrix (C.1)read:

$$\Gamma\big(|p;0,0,n_3,...,1_{\alpha+3},..\rangle,\ g(\alpha,\beta=0,\gamma=0),\ \alpha\big) = \big\{(\cos\alpha-\sin\alpha)\mathbf{e}_1 + (\sin\alpha+\cos\alpha)\mathbf{e}_2 + \xi_3\sqrt{n_3+1}\mathbf{e}_3 \mid \alpha\in\mathbf{R}, \xi_3=\pm1, \big\}. \tag{C.7}$$

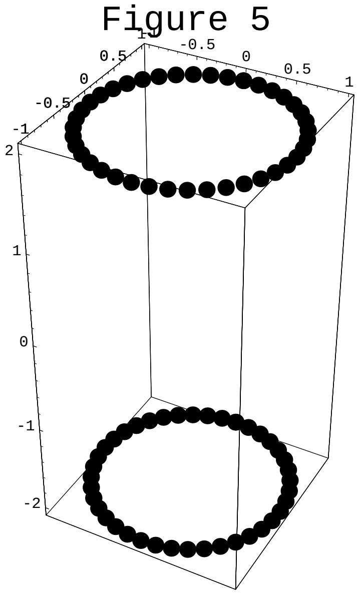
It takes some time to show that the nests corresponding to the 1B and 1C solutions of (6.85) whenever the system is in the state $|p;0,0,n_3,...,1_{\alpha+3},..\rangle$ coincide with (C.7). Therefore the nests of the αth particle corresponding to the state $|p;0,0,n_3,...,1_{\alpha+3},..\rangle$ are

$$\Gamma\big(|p;0,0,n_3,...,1_{\alpha+3},..\rangle,\ \alpha\big) = \big\{(\cos\alpha-\sin\alpha)\mathbf{e}_1 + (\sin\alpha+\cos\alpha)\mathbf{e}_2 + \xi_3\mathbf{e}_3 \mid \alpha\in\mathbf{R}, \xi_3=\pm1, \big\}. \tag{C.8}$$

This proves the third part (6.91)of proposition 6.8.

For an illustration set $n_3 = 3$. The corresponding nests (C.8) of αth particle in the state $|p;0,0,3_3,...,1_{\alpha+3},..\rangle$ are indicated symbolically on Figure 3 as small black balls. There are infinitely many of them situated on two circles with radius $\sqrt{2}$ around the $z-$axes, which are on a distance 2 above and below the $x0y$ plane.

Figure 5



The space distribution of the nests of the αth particle, corresponding to an arbitrary state $|p;0,0,n_3,...,1_{\alpha+3},..\rangle$, is similar: there are infinitely many nests situated on two circles with radius $\sqrt{2}$ around the $z-$axes, which are on a distance $\sqrt{n_3+1}$ above and below the $x0y$ plane.

In a similar way one derives that the nests of the rest of the basis states from Class III are (6.89) and (6.90).

Appendix D: Angular momentum structure of $V(N,p)$ for p=1,2,3.

The case p=1. According to conclusion (7.21) the state space $V(N,p=1)$ can be represented as follows:

$$\begin{aligned} V(N,p=1) = {} & V_1(N,p=1,n_b=0)\otimes V_2(N,p=1,n_f=1), \\ & \oplus V_1(N,p=1,n_b=1)\otimes V_2(N,p=1,n_f=0). \end{aligned}$$

In this case $V_1(N,p=1,n_b=0)$ and $V_1(N,p=1,n_b=1)$ are irreducible with respect to the rotation group. Therefore

$$\begin{aligned} V(N,p=1) = {} & V_1(N,p=1,n_b=0,S=0)\otimes V_2(N,p=1,n_f=1), \\ & \oplus V_1(N,p=1,n_b=1,S=1)\otimes V_2(N,p=1,n_f=0). \end{aligned} \tag{D.1}$$

The case p=2. Again from (7.21)

$$\begin{aligned} V(N,p=2) = {} & \Theta(N-2)V_1(N,p=2,n_b=0)\otimes V_2(N,p=2,n_f=2) \\ & \oplus V_1(N,p=2,n_b=1)\otimes V_2(N,p=2,n_f=1) \\ & \oplus V_1(N,p=2,n_b=2)\otimes V_2(N,p=2,n_f=0.) \end{aligned}$$

This time the last subspace above is $SO(3)$ reducible. Therefore for the angular momentum content of $V(N,p=2)$ we obtain:

$$\begin{aligned} V(N,p=2) = {} & \Theta(N-2)V_1(N,p=2,n_b=0,S=0)\otimes V_2(N,p=2,n_f=2) \\ & \oplus V_1(N,p=2,n_b=1,S=1)\otimes V_2(N,p=2,n_f=1) \\ & \oplus V_1(N,p=2,n_b=2,S=2)\otimes V_2(N,p=2,n_f=0) \\ & \oplus V_1(N,p=2,n_b=2,S=0)\otimes V_2(N,p=2,n_f=0). \end{aligned} \tag{D.2}$$

The case p=3. This time

$$\begin{aligned} V(N,p=3) = &\Theta(N-3)V_1(N,p=3,n_b=0)\otimes V_2(N,p=3,n_f=3)\\ &\oplus \Theta(N-2)V_1(N,p=3,n_b=1)\otimes V_2(N,p=3,n_f=2)\\ &\oplus V_1(N,p=3,n_b=2)\otimes V_2(N,p=3,n_f=1)\\ &\oplus V_1(N,p=3,n_b=3)\otimes V_2(N,p=3,n_f=0). \end{aligned}$$

The decomposition of each V_1 into irreducible $SO(3)$ modules yields (see (7.24):

$$\begin{aligned} V(N,p=3) = &\Theta(N-3)V_1(N,p=3,n_b=0,S=0)\otimes V_2(N,p=3,n_f=3) && (D.3a)\\ &\oplus \Theta(N-2)V_1(N,p=3,n_b=1,S=1)\otimes V_2(N,p=3,n_f=2) && (D.3b)\\ &\oplus V_1(N,p=3,n_b=2,S=2)\otimes V_2(N,p=3,n_f=1) && (D.3c)\\ &\oplus V_1(N,p=3,n_b=2,S=0)\otimes V_2(N,p=3,n_f=1) && (D.3d)\\ &\oplus V_1(N,p=3,n_b=3,S=3)\otimes V_2(N,p=3,n_f=0) && (D.3e)\\ &\oplus V_1(N,p=3,n_b=3,S=1)\otimes V_2(N,p=3,n_f=0) && (D.3f) \end{aligned}$$

Since N is the same for all basis vectors, instead of $||N,p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle$ we write $||p,n_b,S,S_3,n_4,...,n_{N+3}\rangle\rangle$). The angular momentum basis vectors in each subspace (D.3a) - (D.3f), expressed via the reduced basis and via the initial basis, read

$$D.3a): ||p,n_b=0,S=0,S_3=0;n_4,..\rangle\rangle = |p;0,0,0;n_4,..) = |p;0,0,0;n_4,..\rangle,\ \ N\neq 1,2;$$

$$\begin{aligned} (D.3b): ||p,n_b=1,S=1,S_3=1;n_4,..\rangle\rangle &= -|p;0,1,0;n_4,..)\\ &= -\frac{1}{\sqrt{2}}|p;1,0,0;n_4,..\rangle - \frac{i}{\sqrt{2}}|p;0,1,0;n_4,..\rangle, \quad N\neq 1\\ ||p,n_b=1,S=1,S_3=0;n_4,..\rangle\rangle &= |p;0,0,1;n_4,..) = |p;0,0,1;n_4,..\rangle;\\ ||p,n_b=1,S=1,S_3=-1;n_4,..\rangle\rangle &= |p;1,0,0;n_4,..)\\ &= \frac{1}{\sqrt{2}}|p;1,0,0;n_4,..\rangle - \frac{i}{\sqrt{2}}|p;0,1,0;n_4,..\rangle, \quad N\neq 1 \end{aligned}$$

$$\begin{aligned} (D.3c): ||p,n_b=2,S=2,S_3=2;n_4,..\rangle\rangle &= |p;0,2,0;n_4,..)\\ &= \frac{1}{2}|p;2,0,0;n_4,..\rangle + \frac{i}{\sqrt{2}}|p;1,1,0;n_4,..\rangle - \frac{1}{2}|p;0,2,0;n_4,..\rangle;\\ ||p,n_b=2,S=2,S_3=1;n_4,..\rangle\rangle &= -|p;0,1,1;n_4,..) \end{aligned}$$

$$= -\frac{1}{\sqrt{2}}|p;1,0,1;n_4,..\rangle - \frac{i}{\sqrt{2}}|p;0,1,1;n_4,..\rangle;$$

$$||p,n_b=2,S=2,S_3=0;n_4,..\rangle\rangle = \sqrt{\frac{2}{3}}|p;0,0,2;n_4,..) - \frac{1}{\sqrt{3}}|p;1,1,0;n_4,..)$$

$$= \sqrt{\frac{2}{3}}|p;0,0,2;n_4,..\rangle - \frac{1}{\sqrt{6}}|p;2,0,0;n_4,..\rangle - \frac{1}{\sqrt{6}}|p;0,2,0;n_4,..\rangle;$$

$$||p,n_b=2,S=2,S_3=-1;n_4,..\rangle\rangle = |p;1,0,1;n_4,..)$$

$$= \frac{1}{\sqrt{2}}|p;1,0,1;n_4,..\rangle - \frac{i}{\sqrt{2}}|p;0,1,1;n_4,..\rangle,;$$

$$||p,n_b=2,S=2,S_3=-2;n_4,..\rangle\rangle = |p;2,0,0;n_4,..)$$

$$= \frac{1}{2}|p;2,0,0;n_4,..\rangle - \frac{i}{\sqrt{2}}|p;1,1,0;n_4,..\rangle + \frac{1}{2}|p;0,2,0;n_4,..\rangle;$$

$$(D.3d): ||p,n_b=2,S=0,S_3=0;n_4,..\rangle\rangle = \frac{1}{\sqrt{3}}|p;0,0,2;n_4,..) + \sqrt{\frac{2}{3}}|p;1,1,0;n_4,..)$$

$$= \frac{1}{\sqrt{3}}|p;0,0,2;n_4,..\rangle + \frac{1}{\sqrt{3}}|p;2,0,0;n_4,..\rangle + \frac{1}{\sqrt{3}}|p;0,2,0;n_4,..\rangle;$$

$$(D.3e): ||p,n_b=3,S=3,S_3=3;n_4,..\rangle\rangle = -|p;0,3,0;n_4,..) = -\frac{\sqrt{2}}{4}|p;3,0,0;n_4,..\rangle$$

$$- \frac{\mathrm{i}\sqrt{6}}{4}|p;2,1,0,n_4,..\rangle + \frac{\sqrt{6}}{4}|p;1,2,0,n_4,..\rangle - \frac{\mathrm{i}\sqrt{2}}{4}|p;0,3,0,n_4,..\rangle;$$

$$||p,n_b=3,S=3,S_3=2;n_4,..\rangle\rangle = |p;0,2,1;n_4,..)$$

$$= \frac{1}{2}|p;2,0,1;n_4,..\rangle + \frac{i}{\sqrt{2}}|p;1,1,1;n_4,..\rangle - \frac{1}{2}|p;0,2,1;n_4,..\rangle;$$

$$||p,n_b=3,S=3,S_3=1;n_4,..\rangle\rangle = \frac{1}{\sqrt{5}}|p;1,2,0;n_4,..) - \sqrt{\frac{4}{5}}|p;0,1,2;n_4,..)$$

$$= \frac{\sqrt{6}}{4\sqrt{5}}|p;3,0,0;n_4,..\rangle + \frac{\mathrm{i}\sqrt{2}}{4\sqrt{5}}|p;2,1,0,n_4,..\rangle + \frac{\sqrt{2}}{4\sqrt{5}}|p;1,2,0;n_4,..\rangle$$

$$- \frac{\mathrm{i}\sqrt{6}}{4\sqrt{5}}|p;0,3,0,n_4,..\rangle - \frac{\sqrt{2}}{\sqrt{5}}|p;1,0,2;n_4,..\rangle + \mathrm{i}\frac{\sqrt{2}}{\sqrt{5}}|p;0,1,2;n_4,..\rangle;$$

$$||p,n_b=3,S=3,S_3=0;n_4,..\rangle\rangle = \sqrt{\frac{2}{5}}|p;0,0,3;n_4,..) - \sqrt{\frac{3}{5}}|p;1,1,1;n_4,..)$$

$$= \sqrt{\frac{2}{5}}|p;0,0,3;n_4,..\rangle - \sqrt{\frac{3}{10}}|p;0,2,1;n_4,..\rangle - \sqrt{\frac{3}{10}}|p;2,0,1;n_4,..\rangle;$$

$$||p,n_b=3,S=3,S_3=-1;n_4,..\rangle\rangle = -\frac{1}{\sqrt{5}}|p;2,1,0;n_4,..) + \sqrt{\frac{4}{5}}|p;1,0,2;n_4,..)$$

$$= -\frac{\sqrt{6}}{4\sqrt{5}}|p;3,0,0;n_4,..\rangle + \frac{\mathrm{i}\sqrt{2}}{4\sqrt{5}}|p;2,1,0,n_4,..\rangle - \frac{\sqrt{2}}{4\sqrt{5}}|p;1,2,0;n_4,..\rangle$$

$$+\frac{\mathrm{i}\sqrt{6}}{4\sqrt{5}}|p;0,3,0,n_4,..\rangle+\frac{\sqrt{2}}{\sqrt{5}}|p;1,0,2;n_4,..\rangle-\mathrm{i}\frac{\sqrt{2}}{\sqrt{5}}|p;0,1,2;n_4,..\rangle;$$

$$||p,n_b=3,S=3,S_3=-2;n_4,..\rangle\rangle=|p;2,0,1;n_4,..)$$
$$=\frac{1}{2}|p;2,0,1;n_4,..\rangle-\frac{i}{\sqrt{2}}|p;1,1,1;n_4,..\rangle+\frac{1}{2}|p;0,2,1;n_4,..\rangle;$$

$$||p,n_b=3,S=3,S_3=-3;n_4,..\rangle\rangle=|p;3,0,0;n_4,..)=\frac{\sqrt{2}}{4}|p;3,0,0;n_4,..\rangle$$
$$-\frac{\mathrm{i}\sqrt{6}}{4}|p;2,1,0,n_4,..\rangle-\frac{\sqrt{6}}{4}|p;1,2,0,n_4,..\rangle+\frac{\mathrm{i}\sqrt{2}}{4}|p;0,3,0,n_4,..\rangle;$$

$$(D.3f):||p,n_b=3,S=1,S_3=1;n_4,..\rangle\rangle=-\frac{1}{\sqrt{5}}|p;0,1,2;n_4,..)-\sqrt{\frac{4}{5}}|p;1,2,0;n_4,..)$$
$$=-\sqrt{\frac{3}{10}}|p;3,0,0;n_4,..\rangle-\frac{\mathrm{i}}{\sqrt{10}}|p;2,1,0,n_4,..\rangle-\frac{1}{\sqrt{10}}|p;1,2,0;n_4,..\rangle$$
$$+\mathrm{i}\sqrt{\frac{3}{10}}|p;0,3,0,n_4,..\rangle-\frac{1}{\sqrt{10}}|p;1,0,2;n_4,..\rangle+\frac{\mathrm{i}}{\sqrt{10}}|p;0,1,2;n_4,..\rangle;$$

$$||p,n_b=3,S=1,S_3=0;n_4,..\rangle\rangle=\sqrt{\frac{3}{5}}|p;0,0,3;n_4,..)+\sqrt{\frac{2}{5}}|p;1,1,1;n_4,..)$$
$$\sqrt{\frac{3}{5}}|p;0,0,3;n_4,..\rangle+\frac{1}{\sqrt{5}}|p;2,0,1;n_4,..\rangle+\frac{1}{\sqrt{5}}|p;0,2,1;n_4,..\rangle;$$

$$||p,n_b=3,S=1,S_3=-1;n_4,..\rangle\rangle=\frac{1}{\sqrt{5}}|p;1,0,2;n_4,..)+\sqrt{\frac{4}{5}}|p;2,1,0;n_4,..)$$
$$=\frac{1}{\sqrt{10}}|p;1,0,2;n_4,..\rangle-\frac{\mathrm{i}}{\sqrt{10}}|p;0,1,2;n_4,..\rangle+\sqrt{\frac{3}{10}}|p;3,0,0;n_4,..\rangle$$
$$-\frac{\mathrm{i}}{\sqrt{10}}|p;2,1,0;n_4,..\rangle+\frac{1}{\sqrt{10}}|p;1,2,0;n_4,..\rangle-\mathrm{i}\sqrt{\frac{3}{10}}|p;0,3,0,n_4,..\rangle.$$

$\frac{1}{2}$, $\sqrt{\frac{4}{5}}$, $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3}}{2}$
$\frac{a\sqrt{b}}{c\sqrt{d}}$, $\frac{1}{\sqrt{10}}$, $\sqrt{\frac{3}{10}}$

Appendix E: Eigenstates and eigenvalues of the momentum operators

The kth projection of the momentum operator for αth particle read (see (6.9b))

$$\hat{p}_{\alpha k}(t)=\mathrm{i}\varepsilon\Big(E_{k,\alpha+3}\mathrm{e}^{\mathrm{i}\varepsilon\omega t}-E_{\alpha+3,k}\mathrm{e}^{-\mathrm{i}\varepsilon\omega t}\Big). \qquad (E.1)$$

In the next proposition we write down the eigenstates of $\hat{p}_{\alpha k}(t)$ and their eigenvalues.

Proposition E.1. *The eigenvectors of the momentum operator* $\hat{p}_{\alpha,k}$, $k = 1, 2, 3$, *read:*

a. Eigenvalue 0 :

$$w^0_{\alpha k}(.., 0_k, .., 0_{\alpha+3,..}) = |p; .., 0_k, ..., 0_{\alpha+3}, ..\rangle, \tag{E.2}$$

b. Eigenvalues $\pm\sqrt{n}_k$ $(n_k \neq 0)$:

$$w^{\pm}_{\alpha k}(.., n_k, .., 0_{\alpha+3,..}) = \frac{1}{\sqrt{2}}\Big(|p; .., n_k, .., 0_{\alpha+3}, ..\rangle \\ \pm\, \mathrm{i}\varepsilon(-1)^{n_1+\ldots+n_{\alpha+2}}\mathrm{e}^{-\mathrm{i}\varepsilon\omega t}|p; .., n_k - 1, .., 1_{\alpha+3}, ..\rangle, \quad n_k > 0, \tag{E.3}$$

The inverse to (E.3) relations take the form (10370):

$$|p; .., n_k, .., 0_{\alpha+3}, ..\rangle = \frac{1}{\sqrt{2}}\Big(w^+_{\alpha k}(.., n_k, ..., 0_{\alpha+3}, ..) + w^-_{\alpha k}(.., n_k, ..., 0_{\alpha+3}, ..)\Big), \\ |p; .., n_k - 1, .., 1_{\alpha+3}, ..\rangle = \frac{\mathrm{i}}{\sqrt{2}}\varepsilon(-1)^{(n_1+\ldots+n_{\alpha+2})}\mathrm{e}^{\mathrm{i}\varepsilon\omega t} \\ \Big(w^-_{\alpha k}(.., n_k, ..., 0_{\alpha+3}, ..) - w^+_{\alpha k}(.., n_k, ..., 0_{\alpha+3}, ..)\Big), \quad n_k > 0. \tag{E.4}$$

The same equations in a compact form:

$$|p; .., n_k, , .., n_{\alpha+3}, ..\rangle = \frac{\mathrm{i}^{n_{\alpha+3}}}{\sqrt{2}}(-1)^{(n_1+\ldots+n_{\alpha+2}+1)n_{\alpha+3}}\mathrm{e}^{\mathrm{i}\varepsilon n_{\alpha+3}\omega t} \\ \Big(w^-_{\alpha k}(.., n_k + n_{\alpha+3}, ..., 0_{\alpha+3}, ..) + (-1)^{n_{\alpha+3}}w^+_{\alpha k}(.., n_k + n_{\alpha+3}, ..., 0_{\alpha+3}, ..)\Big), \tag{E.5}$$

Note that the eigenvectors of the momentum operators $\hat{p}_{\alpha,k}$ and the eigenvectors of the position operators $\hat{r}_{\alpha,k}$, corresponding to zeroth eigenvalues coincide,

$$w^0_{\alpha k}(.., 0_k, .., 0_{\alpha+3,..}) = v^0_{\alpha k}(.., 0_k, .., 0_{\alpha+3,..}) = |p; .., 0_k, ..., 0_{\alpha+3}, ..\rangle. \tag{E.6}$$

Therefore $\hat{r}_{\alpha,k}$ and $\hat{p}_{\alpha,k}$ commute on the subspace spanned by all states $|p; .., 0_k, .., 0_{\alpha+3}, ..\rangle$

8. Concluding remarks

We have studied the properties of $N-$particle noncanonical harmonic oscillator, considering it as a Wigner quantum system with the additional requirement the position and the momentum operators of the oscillating particles to be odd operators, generating the Lie superalgebra $sl(3|N)$.

The idea for such a requirement is a natural one if one takes into account that the canonical PM-operators generate also a representation of a LS, but from the class $\mathcal{B}$, namely the orthosymplectic LS $osp(1|6N)$. We should admit however that such an assumption is of pure mathematical origin and as such the $sl(3|N)$ oscillator is essentially a mathematical model. Nevertheless it is surprising to see how rich is the idea of Wigner to relax the postulates of QM replacing the postulate about the CCRs with the requirement both the Heisenberg and the Hamiltonian's equations to be fulfilled simultaneously.

We see that despite of the circumstance that the equations of motion (1.5), the Heisenberg equations (1.6) and the Hamiltonian (1.1) are formally the same as for N free oscillators, the properties of the $sl(3|N)$ oscillator are very different from those of the corresponding canonical such oscillator.

On the first place Conclusion 5.9 tell us that there exist strong space correlations between the particles. In particular all particles from Class II, which have one and the same fermionic coordinates "share" 8 common nests independently on the number N of the "inhabitants", the oscillating particles. Clearly these correlations are of statistical origin.

Secondly, and this is an essentially new result, there exists even stronger statistical correlation between the angular momenta of the particles: the components of the angular momentum of all N particles $\hat{M}_{\alpha 1},\ \hat{M}_{\alpha 2},\ \hat{M}_{\alpha 3}$ coincide, they do not depend on the label of the particle α, see (3.30). Consequently all particles have one and the same angular momentum. For instance if the system is in a reduce basis state $||N, p, b, S, S_3, n_4, n_5, ..., n_{N+3}\rangle\rangle$, see (7.28), then all particles have one and the same angular momentum $\mathbf{S}$ and one and the same projection along $z-$axes S_3.

Another property to mention is the space structure of the basis states $|p; n\rangle$. Typically each such state corresponds to a picture when each oscillating particle is measured to occupy with equal probability only finite number of points, as a rule the eight vertices of a parallelepiped. As a result the entire oscillator is confined in the coordinate (and the momentum) space, it is "locked" within a sphere with a finite radius. This property is in the origin of the relations $\Delta\hat{r} \leq \sqrt{p}, \quad \Delta\hat{p} \leq \sqrt{p}$. Therefore in the limit $\hbar \to\ 0$ the entire system collapses into a point.

We should not forget to point out again that all our conclusions and results hold only for the Hamiltonian (1.1). Is the idea of Wigner applicable for another Hamiltonians and what are their predictions is an open question. A first example in this directions was given in [20] for a magnetic dipole precessing in magnetic field

The results obtained in the present paper are based on a pure class of irreducible representations of $sl(3|N)$, which are numbered by only one positive integer p. The reason for such a choice stems from the observation that explicit expressions for all finite-dimensional representations of the LS $sl(3|N)$ does not exists so far.

Therefore a natural next step would be to extend the class of representations of the underlying superalgebra. The simplest way to do this is to consider the realizations of the superalgebra $sl(3|N)$ in the Fock space of three pairs of Fermi CAOs (even generators) and N pairs of Bose CAOs (odd generators), where again the Bose CAOs anticommute with the Fermi operators. This realization will put strong limitations on the angular momentum of the entire system. Another possibility is to consider the Holstein-Primakoff realization of $gl(3|N+1)$ [N 119]. In this case the Bose operators are even operators and the Fermi CAOs are odd and Bose operators commute with Fermi CAOs. In such a case the Fock space wouldn't be anymore an irreducible module of an orthosymplectic LS but of the more familiar $gl(3|N+1)$.

As already mentioned, the main problem arising in the context of WQS is to determine the common solutions of the equations of motion (1.5) and of the Heisenberg equations (1.6), which satisfy the defining postulates (P1) - (P6). Stated in this way, the problem does not require the PM-operators to be elements of a Lie superalgebra, of a Lie algebra, or of any another algebraic structure. From this point of view the LS $gl(3|N+1)$ was only a tool to find at least some solutions of the problem. And these solutions turned to predict WQS with interesting properties.

Are there any indications that WQSs exist in nature? Can they be of real interest in physics? In this relation we mention that recently the finite-level quantum systems become of great interest in quantum computing. And such are the WQSs. The properties of the WQSs reported in the present paper resemble also the artificial atoms arising in condensed matter physics [61] and more generally various kinds of clusters (see, for instance [62]). Is there any deeper connection behind just a resemblance? That is what we would like to know too.

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